DIAMONDS ON TREES

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ABSTRACT. We generalize the diamond principle and its variants using the notion of stationarity in trees introduced by Brodsky in [5] and [4]. In particular, we show that if T is a nonspecial ω_1-tree, then $\Diamond_T \implies \Diamond$, and if T is a Suslin tree, then $\Diamond_T \iff \Diamond$. We also prove that $\Diamond^*$ implies $\Diamond_T$ (yielding the consistency of $\Diamond_T$) and establish the consistency of $\neg\Diamond^* + (\forall T \text{ nonspecial } \omega_1\text{-tree } (\Diamond_T))$. Finally, we demonstrate that it is consistent with $\Diamond$ that there exists a nonspecial ω_1-tree with $\neg\Diamond_T$, introducing two forcing properties—$\sigma(S)$-closed and strategically closed in models—which are preserved under countable support iterations.

1. INTRODUCTION

In [35] and [36], Stevo Todorcevic introduced a new notion of stationarity for subsets of ω_1 parameterized by an arbitrary nonspecial tree of height ω_1. He proved that such a tree T is special if and only if there exists a regressive map $f\colon T \to T$ such that $f^{-1}(\{t\})$ is special for every $t \in T$. This notion was later used by Ari Brodsky in [4] and [5] to define a notion of nonstationarity for subsets of T itself[1], rather than for subsets of its height ω_1. Thus, a subset $X \subseteq T$ is nonstationary if and only if there is a regressive map $f\colon X \to T$ such that $f^{-1}(\{t\})$ is special for all $t \in T$.

Once the notion of stationarity is in place, it becomes natural to define diamond sequences on trees. More precisely, one considers sequences of the form $(D_t \mid t \in T)$, where for each node $t \in T$, the set D_t is a subset of $\{s \in T \mid s < t\}$. These sequences are required to *guess* every subset $X \subseteq T$ in a stationary way. We denote such sequences by $\Diamond_T$. In this work, we study $\Diamond_T$-sequences in depth and explore the relationships between these and the usual diamond principles defined on ω_1.

Our main results include the following. We show that if T is an ω_1-tree that either has a cofinal branch or is almost-Suslin (in particular, if T is Suslin), then $\Diamond_T$ is equivalent to the classical diamond principle $\Diamond$. This gives rise to the informal intuition that for Aronszajn trees, the *thinner* the tree, the more $\Diamond_T$ resembles $\Diamond$. On the other hand, we prove that $\Diamond_{2^{<\omega_1}}$ holds in ZFC. This gives rise to the informal intuition that the more cofinal branches a tree T has, the more likely it is that $\Diamond_T$ holds[2].

We also prove that $\Diamond^*$ implies $\Diamond_T$ for every nonspecial and well-pruned ω_1-tree T; in particular, we prove an analogue of Kunen's theorem stating that the principles $\Diamond$ and $\Diamond^-$ are equivalent; then we show that $\Diamond^*$ turns out to be strictly stronger than the principle asserting that $\Diamond_T$ holds for every nonspecial ω_1-tree T.

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[1]Brodsky's motivation was to extend the well-known Balanced Baumgartner–Hajnal–Todorcevic Theorem to the setting of trees.

[2]Curiously, the complete opposite is true for another diamond principle for trees, $\Diamond(T)$, studied in [20].

We also establish the consistency of $\Diamond$ together with the failure of $\Diamond_T$ for some nonspecial Aronszajn tree T. This is achieved by starting from a specific nonspecial Aronszajn tree in a model of $V = L$, and by showing that in a suitable forcing extension $\Diamond$ is preserved while $\Diamond_T$ is destroyed (without specializing T). Hence, for some Aronszajn trees $\Diamond_T$ lies strictly between $\Diamond$ and $\Diamond^*$, demonstrating that $\Diamond_T$ is a genuinely new principle.

We view $\Diamond_T$ as an *invariant* for studying nonspecial ω_1-trees, in the sense that it behaves monotonically between trees, that is, if there is a strictly increasing map $f : S \to T$, then $\Diamond_S$ implies $\Diamond_T$ (see part (ii) of Theorem 3.17). In other words, our study of $\Diamond_T$-sequences is not primarily aimed at finding applications, but rather at using them as invariants to better understand the trees themselves. Something that supports this paradigm for studying trees is that through $\Diamond_T$ we can consider three types of nonspecial trees of height ω_1: those for which $\Diamond_T$ holds in ZFC (such as $2^{<\omega_1}$ (see Theorem 3.28)), those for which $\Diamond_T$ is equivalent to the classical $\Diamond$ (such as Suslin trees; see Corollary 4.14), and those for which $\Diamond_T$ is strictly stronger than the classical $\Diamond$ (such as the one exhibited in Theorem 6.1).

The article is organized as follows. In Section 2, we compile all the necessary definitions and results from Brodsky's work on stationarity in trees (without proofs), and we include a few additional results that will be needed later and that are not explicitly found in his papers. In Section 3, we introduce the concept of $\Diamond_T$ and investigate some of its basic properties; we start by studying how $\Diamond_S$ and $\Diamond_T$ compare when S is a subtree of T, then we prove that $\Diamond_T$ imposes certain cardinal restrictions on the size of T in the same way that $\Diamond$ implies CH, that $\Diamond_{2^{<\omega_1}}$ holds in ZFC, and that $\Diamond_T$ implies $\Diamond$ for every nonspecial ω_1-tree. We conclude the section by showing that $\Diamond_T$ can be expressed within a more familiar framework, commonly used to define the diamond principle on $P_\kappa(\lambda)$. In Section 4, we prove the consistency of the statement $\forall T \left(T \text{ is a nonspecial } \omega_1\text{-tree} \implies \Diamond_T \right)$ and we show that $\Diamond_T \iff \Diamond$ when T is an almost-Suslin tree. In Section 5 we construct a model of $\neg\Diamond^*$ where $\Diamond_T$ holds for every T nonspecial ω_1-tree. In Section 6 we separate the notion of $\Diamond_T$ from the classical diamond principle $\Diamond$ for some Aronszajn tree T. In this section we introduce the two forcing properties —namely, the $\sigma(S)$-closed and the strategically closed in models forcings—and prove iteration theorems for both. Finally we conclude with a section of open questions.

2. Preliminaries and Brodsky's notion of stationary for trees

We begin by recalling some classical guessing principles that will appear throughout this article, starting with Jensen's notion of diamond:

Definition 2.1 ([17])**.** Let $S \subseteq \omega_1$ be stationary. A sequence $(D_\alpha)_{\alpha \in S}$ is a $\Diamond(S)$-*sequence* if

(1) $D_\alpha \subseteq \alpha$ for every $\alpha \in S$, and
(2) for every $X \subseteq \omega_1$, the set $\{\alpha \in S \mid X \cap \alpha = D_\alpha\}$ is stationary.

Two related principles that strengthen $\Diamond(S)$ are:

Definition 2.2 ([18])**.** Let $S \subseteq \omega_1$ be stationary. A sequence $(\mathcal{D}_\alpha)_{\alpha \in S}$ is a $\Diamond^*(S)$-*sequence* if

(1) $\mathcal{D}_\alpha \subseteq \mathcal{P}(\alpha)$ and $|\mathcal{D}_\alpha| \leq \omega$ for every $\alpha \in S$, and

(2) for every $X \subseteq \omega_1$, the set $\{\alpha \in S \mid X \cap \alpha \in \mathcal{D}_\alpha\}$ contains a club.

Definition 2.3 ([18])**.** Let $S \subseteq \omega_1$ be stationary. A sequence $(\mathcal{D}_\alpha)_{\alpha \in S}$ is a $\diamondsuit^+(S)$-*sequence* if

(1) $\mathcal{D}_\alpha \subseteq \mathcal{P}(\alpha)$ and $|\mathcal{D}_\alpha| \leq \omega$ for every $\alpha \in S$, and

(2) for every $X \subseteq \omega_1$, there is a club $C \subseteq \omega_1$ such that for all $\alpha \in C \cap S$ we have $X \cap \alpha \in \mathcal{D}_\alpha$ and $C \cap \alpha \in \mathcal{D}_\alpha$.

It is clear that $\diamondsuit^+(S) \implies \diamondsuit^*(S)$; a natural weakening of $\diamondsuit^*(S)$ is the following:

Definition 2.4. Let $S \subseteq \omega_1$ be stationary. A sequence $(\mathcal{D}_\alpha)_{\alpha \in S}$ is a $\diamondsuit^-(S)$-*sequence* if

(1) $\mathcal{D}_\alpha \subseteq \mathcal{P}(\alpha)$ and $|\mathcal{D}_\alpha| \leq \omega$ for every $\alpha \in S$, and

(2) for every $X \subseteq \omega_1$, the set $\{\alpha \in S \mid X \cap \alpha \in \mathcal{D}_\alpha\}$ is stationary.

A remarkable result of Kunen is that $\diamondsuit(S)$ and $\diamondsuit^-(S)$ are equivalent [21, Theorem 7.14]. Consequently, $\diamondsuit^+(S) \implies \diamondsuit^*(S) \implies \diamondsuit(S)$, and none of these implications can be reversed (see [6] and [7]).

Now, we present the definitions and results from Brodsky's work that are needed for the development of this article, for this we will fix some notation for trees and ideals.

Recall that a tree is a partial order $(T, <_T)$ such that for every $t \in T$, the set of predecessors of t is well-ordered by $<_T$. Throughout, we assume every tree has a root, which we identify—without loss of generality—with the empty set $\emptyset$.

Notation 2.5. Let T be a tree.

(1) For each $t \in T$,
$$t\!\downarrow = \{\, s \in T \mid s <_T t \,\}, \quad t\!\uparrow = \begin{cases} \{\, s \in T \mid t <_T s \,\}, & t \neq \emptyset, \\ T, & t = \emptyset. \end{cases}$$

(2) For $t \in T$, $\mathrm{ht}(t)$ is the *height* of t, i.e., the order type of $t\!\downarrow$.

(3) For $S \subseteq T$, $\widehat{S} = \{\, \mathrm{ht}(s) \mid s \in S \,\}$.

(4) For a set of ordinals C, $T \restriction C = \{\, t \in T \mid \mathrm{ht}(t) \in C \,\}$.

(5) For an ordinal α, $T_\alpha = T \restriction \{\alpha\}$, $T_{<\alpha} = T \restriction \alpha$ and $T_{\geq \alpha} = T \setminus T_{<\alpha}$.

The *height* of T is the least ordinal α with $T_\alpha = \emptyset$, denoted by $\mathrm{ht}(T)$. A κ-*tree* is a tree of height κ in which each level T_α has size $< \kappa$. Thus an ω_1-tree has height ω_1 and each level is at most countable.

A *cofinal branch* in T is a chain $B \subseteq T$ meeting every level: $B \cap T_\alpha \neq \emptyset$ for all $\alpha < \mathrm{ht}(T)$. An *antichain* is a subset $A \subseteq T$ in which no two distinct elements are comparable: if $s \neq t \in A$ then neither $s <_T t$ nor $t <_T s$. An ω_1-tree is *Aronszajn* if it has no cofinal branches, and *Souslin* if it has neither cofinal branches nor uncountable antichains. For a tree T of height κ, we say it is *well-pruned* if for every $\alpha < \beta < \kappa$ and every $s \in T_\alpha$ there exists $t \in T_\beta$ with $s <_T t$. We say T is *Hausdorff* if for all distinct $s, t \in T$, if $\mathrm{ht}(s)$ and $\mathrm{ht}(t)$ are limit ordinals, then $s\!\downarrow \neq t\!\downarrow$. Finally, if T is a tree, $X \subseteq T$ and $f : X \to T$, then f is *regressive* if $f(t) <_T t$ for every $t \in X \setminus \{\emptyset\}$ (see [35, Section 1]).

Definition 2.6. Let X be a set. An *ideal* on X is a nonempty family $\mathcal{I} \subseteq \mathcal{P}(X)$[3] such that:

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[3]$\mathcal{P}(X)$ denotes the power set of X, that is, the family of all subsets of X.

(1) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

(2) If $A \subseteq B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$.

For an ideal $\mathcal{I}$ on X its *positive* part is $\mathcal{I}^+ = \{A \subseteq X \mid A \notin \mathcal{I}\}$ while and its *dual* is $\mathcal{I}^* = \{A \subseteq X \mid X \setminus A \in \mathcal{I}\}$. We call an ideal $\mathcal{I}$ *proper* if $X \notin \mathcal{I}$; note that $\mathcal{I}$ is proper exactly when $\mathcal{I}^* \cap \mathcal{I} = \emptyset$, equivalently, when $\mathcal{I}^* \subseteq \mathcal{I}^+$. If $Y \subseteq X$, then the restriction $\mathcal{I} \restriction Y$ is defined by $\mathcal{I} \restriction Y = \{A \cap Y \mid A \in \mathcal{I}\}$ (equivalently, $\mathcal{I} \restriction Y = \{A \in \mathcal{I} \mid A \subseteq Y\}$); it is easy to see that $\mathcal{I} \restriction Y$ is an ideal on Y and it is proper if and only if $Y \in \mathcal{I}^+$. We say that $\mathcal{I}$ is κ-*complete*, where κ is a cardinal, if $\bigcup \mathcal{F} \in \mathcal{I}$ for every $\mathcal{F} \subseteq \mathcal{I}$ with $|\mathcal{F}| < \kappa$.

The rest of the terminology we follow is standard and largely agrees with Brodsky's. The only difference is purely linguistic: in Brodsky's approach, every subset of a tree is called a "subtree", whereas we reserve that term for subsets that are closed under initial segments. Thus, if T is a tree and $S \subseteq T$, we say that S is a *subtree* of T, if whenever $s, t \in T$ satisfy $s \leq_T t$ and $t \in S$, then $s \in S$.

Before defining nonstationary subsets of a tree, we first recall the notion of special subsets:

Definition 2.7. Let T be a tree of height ω_1 and let $U \subseteq T$. We say that U is *special* if U can be written as the union of at most ω antichains; equivalently, there exists a function

$$f : U \longrightarrow \omega$$

such that for all $t, u \in U$, $t <_T u$ implies $f(t) \neq f(u)$.

Note that the collection of special subsets of T forms an ideal on T that is, in fact, ω_1-complete. Also, it is important to observe that a special subset $U \subseteq T$ need *not* be a subtree; that is, U is *not* required to be closed under initial segments.

Special sets are natural analogues of the bounded subsets of ω_1 in the sense that, when ω_1 is viewed as a tree, the ideal of special subtrees coincides with the ideal of bounded (i.e., at most countable) subsets.

Now, we can explicitly formulate Todorcevic's notion of nonstationarity parametrized by a tree T of height ω_1:

Definition 2.8. [35, 36] Let T be a tree of height ω_1. The ideal NS_T on ω_1 is defined as follows: for $X \subseteq \omega_1$ we have $X \in NS_T$ if and only if there exists a regressive function $f \colon T \restriction X \to T$ such that $f^{-1}(\{t\})$ is special for every $t \in T$.

And its natural generalization, due to Brodsky, which motivated our investigation of guessing principles on trees:[4]

Definition 2.9. [4, 5] Let T be a tree of height ω_1. The ideal NS^T on T is defined as follows: for $B \subseteq T$ we have $B \in NS^T$ if and only if there exists a regressive function $f \colon B \to T$ such that $f^{-1}(\{t\})$ is special for every $t \in T$. In case $B \in NS^T$, we say that B is *nonstationary in T*.

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[4]Actually, Todorcevic treats trees of any regular uncountable height, while Brodsky does so for any successor cardinal. Here we restrict to the case of ω_1, as it is the context of this article.

Clearly every special subset of T is nonstationary in T. Additionally, Definition 2.9 generalizes Definition 2.8 in the sense that, for a tree T of height ω_1 and $X \subseteq \omega_1$, we have

$$X \in NS_T \iff T \restriction X \in NS^T.$$

Note also that when $T = \omega_1$, we have $NS^T = NS_T$ and, by Fodor's Lemma [10], both coincide with the classical nonstationary ideal on ω_1.

For many arguments that follow, it is convenient to have an equivalent description of NS^T in terms of *diagonal unions*[5].

Definition 2.10. Let T be a tree of height ω_1. For any collection of subsets of T indexed by the nodes of T, i.e., $(A_t)_{t \in T}$, its diagonal union is defined as:

$$\nabla_{t \in T} A_t = \bigcup_{t \in T} (A_t \cap t\uparrow).$$

Remark 2.11. [4, Lemma 37] Let T be a tree of height ω_1 and $(A_t)_{t \in T}$ a sequence of subsets of T. Then $\nabla_{t \in T} A_t = \{w \in T \mid w \in A_\emptyset \cup \bigcup_{t <_T w} A_t\}$.

The following theorem collects several key facts about the ideal NS^T that will be used throughout the paper.

Theorem 2.12. *Let T be a tree of height ω_1. Then:*

(I) *For every $B \subseteq T$, $B \in NS^T$ if and only if B can be written as a diagonal union of special subsets of T. [4, Lemmas 42 and 45] (cf. [2, p. 9])*

(II) *NS^T is ω_1-complete. [4, Lemma 46]*

(III) *NS^T is closed under diagonal unions; namely, if $\{A_t \mid t \in T\} \subseteq NS^T$, then $\nabla_{t \in T} A_t \in NS^T$. [4, Theorem 47]*

(IV) *If $X \in (NS^T)^+$ and $f : X \to T$ is regressive, then there exists $Y \in (NS^T)^+ \cap \mathcal{P}(X)$ such that $f \restriction Y$ is constant. [4, Theorem 47 and Corollary 44]*

(V) *T is nonspecial if and only if NS^T is a proper ideal if and only if NS_T is a proper ideal. [4, Theorem 49][35, 36]*

Additionally, the following theorem summarizes some of the most important results relating the ideal NS^T to the usual ideal of nonstationary sets in ω_1.

Theorem 2.13. [4, Lemma 51][35, 36] *Let T be a tree of height ω_1 and $X, C \subseteq \omega_1$. Then:*

(i) *If $|X| \leq \omega$ the $T \restriction X \in NS^T$.*

(ii) *If X is a nonstationary subset of ω_1, then $X \in NS_T$ and therefore $T \restriction X \in NS^T$.*

(iii) *In particular, the set of successor nodes of T is a nonstationary subset of T.*

(iv) *If C is a club subset of ω_1, then $T \restriction C \in (NS^T)^*$ and $C \in (NS_T)^*$.*

(v) *If T is a nonspecial tree and C is a club subset of ω_1, then $T \restriction C \notin NS^T$.*

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[5]Actually, Brodsky's original formulation of the ideal NS^T is in terms of diagonal unions.

From now on, the results in this section are no longer (at least not in an obvious manner) found in Brodsky's work. Therefore, even though these results are straightforward, we have decided to include their proofs.

In the remainder of this section, for a tree T and $S \subseteq T$, whenever $t \in S$ we denote by $(t{\uparrow})_S$ the set $(t{\uparrow}) \cap S$, and analogously with $t{\downarrow}_S$. Also, in the following proofs we use the fact that if $X \subseteq S$, then X is special in S if and only if it is special in T, since S inherits its order from T.

Proposition 2.14. *Let T be a tree of height ω_1 and $S \subseteq T$, then $NS^S \subseteq (NS^T) \restriction S$.*[6]

Proof. Let $(X_s)_{s \in S}$ be such that each X_s is a special subset of S. Now for every $t \in T$ define $Y_t = X_t$ in the case that $t \in S$ and $Y_t = \emptyset$ otherwise; note that $Y_t \subseteq S$ for all $t \in T$.

Claim 2.15. $(\nabla_{t \in T} Y_t) \cap S = \nabla_{s \in S} X_s$.

Proof of the Claim. ($\subseteq$) If $u \in (\nabla_{t \in T} Y_t) \cap S$ there there is $t \in T$ such that $u \in Y_t \cap (t{\uparrow})_T$, then $Y_t \neq \emptyset$ and consequently $t \in S$, then $u \in X_t \cap (t{\uparrow})_T$, but as $u \in S$ then $u \in (t{\uparrow})_S$, then $u \in X_t \cap (t{\uparrow})_S \subseteq \nabla_{s \in S} X_s$.

($\supseteq$) If $u \in \nabla_{s \in S} X_s$, then there is $s \in S$ such that $u \in X_s \cap (s{\uparrow})_S$, then $u \in X_s \cap (s{\uparrow})_T$ as $(s{\uparrow})_S \subseteq (s{\uparrow})_T$, then $u \in \nabla_{t \in T} Y_t$. $\qquad\qquad$ $_{Claim}\square$ $\qquad\qquad\qquad$ $\square$

Recall that $S \subseteq T$ is a subtree of T if it is closed under initial segments.

Lemma 2.16. *Let T be a tree of height ω_1. If S is a subtree of T, then $NS^S = NS^T \restriction S$.*

Proof. By Proposition 2.14, it is enough to prove that $NS^T \restriction S \subseteq NS^S$. Let $(X_t)_{t \in T}$ be a sequence of special subsets. For each $s \in S$, let $Y_s = X_s \cap S$. Clearly, each Y_s is special, as it is a subset of a special set.

Claim 2.17. $\nabla_{s \in S} Y_s = (\nabla_{t \in T} X_t) \cap S$.

Proof of the Claim. ($\subseteq$) Let $u \in Y_s \cap (s{\uparrow})_S$ for some $s \in S$, then $u \in X_s \cap (s{\uparrow})_T$ and thus $u \in \nabla_{t \in T} X_t$.

($\supseteq$) Let $u \in (\nabla_{t \in T} X_t) \cap S$. Thus, by Remark 2.11 we have $u \in X_\emptyset$ or $\exists t \in T(t < u \land u \in X_t)$. If $u \in X_\emptyset$, then $u \in Y_\emptyset = X_\emptyset \cap S$. If $\exists t \in T(t < u \land u \in X_t)$, then $t \in S$ as S is a subtree of T, then $u \in X_t \cap (t{\uparrow})_S = Y_t \cap (t{\uparrow})_S \subseteq \nabla_{s \in S} Y_s$. $\qquad$ $_{Claim}\square$ $\qquad\qquad\qquad$ $\square$

Note that part (V) of Theorem 2.12 together with Lemma 2.16 implies the following.

Corollary 2.18. *If T is a tree of height ω_1 and S is a subtree of T, then S is special if and only if $S \in NS^T$.*

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[6]Here we view S with the order inherited from T; hence, S is also a tree (although it is not a subtree of T) and it makes sense to consider NS^S (making the convention $NS^S = \mathcal{P}(S)$ in case $\mathrm{ht}(S) \leq \omega$).

3. Diamonds on trees and its basic properties

In light of Definition 2.9 and the fact that it includes the case $T = \omega_1$, the following guessing principles are natural generalizations of the usual ones and represent the central objects of study in this article:

Definition 3.1. If T is a nonspecial tree of height ω_1. A sequence $(D_t \mid t \in T)$ is called a $\diamondsuit_T$-sequence if:

(1) $D_t \subseteq t{\downarrow}$ for every $t \in T$.
(2) For every $X \subseteq T$ we have $\{t \in T \mid X \cap t{\downarrow} = D_t\} \notin NS^T$.

We refer to condition (2) as *every set X is guessed in a stationary set.*

Definition 3.2. If T is a nonspecial tree of height ω_1. A sequence $(\mathcal{D}_t \mid t \in T)$ is called a $\diamondsuit_T^*$-sequence if:

(1) $\mathcal{D}_t \subseteq \mathcal{P}(t{\downarrow})$ and $|\mathcal{D}_t| \leq \omega$ for every $t \in T$.
(2) For every $X \subseteq T$ we have $\{t \in T \mid X \cap t{\downarrow} \notin \mathcal{D}_t\} \in NS^T$.

As usual, the existence of a $\diamondsuit_T$-sequence is denoted simply by $\diamondsuit_T$, and similarly for $\diamondsuit_T^*$. As we will see later—and in analogy with the classical case—$\diamondsuit_T^*$ implies $\diamondsuit_T$.

We begin the study of $\diamondsuit_T$ by showing that, for ω_1-trees, it is at least as strong as the classical $\diamondsuit$. However, for our later work—where we separate $\diamondsuit_T$ from $\diamondsuit$—it will be useful to prove a somewhat stronger statement, one whose immediate corollary is $\diamondsuit_T \implies \diamondsuit$.

Theorem 3.3. *Let T be a nonspecial ω_1-tree and $S \subseteq \omega_1$ such that $T \restriction S \in NS^T$, then $\diamondsuit_T \implies \diamondsuit(\omega_1 \setminus S)$.*

Proof. Let $(D_t)_{t \in T}$ be a $\diamondsuit_T$-sequence. Now, for every $\alpha \in \omega_1$, define $A_\alpha = \{\widehat{D_t} \mid \mathrm{ht}(t) = \alpha\}$. Note that $A_\alpha \subseteq \mathcal{P}(\alpha)$ and $|A_\alpha| \leq \omega$ for every $\alpha \in \omega_1$.

Claim 3.4. $(A_\alpha)_{\alpha \in \omega_1 \setminus S}$ is a $\diamondsuit_{\omega_1 \setminus S}^-$-sequence.

Proof of the Claim. Let $X \subseteq \omega_1$, and note that $\{t \in T \mid (T \restriction X) \cap t{\downarrow} = D_t\} \notin NS^T$ since $(D_t)_{t \in T}$ is a $\diamondsuit_T$-sequence. Now, let $E := \{t \in T \mid (T \restriction X) \cap t{\downarrow} = D_t\} \setminus (T \restriction S)$ and note that $E \notin NS^T$. Thus, $\widehat{E} \subseteq \{\alpha \in \omega_1 \setminus S \mid X \cap \alpha \in A_\alpha\}$. Indeed, if $\alpha \in \widehat{E}$, then $\alpha \in \omega_1 \setminus S$ and there is $t \in E$ such that $\mathrm{ht}(t) = \alpha$, so $(T \restriction X) \cap t{\downarrow} = D_t$, and thus $X \cap \alpha = \widehat{D_t} \in A_\alpha$. Finally, $\widehat{E}$ is stationary in ω_1, since otherwise $T \restriction \widehat{E} \in NS^T$ by (ii) of Theorem 2.13, but $E \subseteq T \restriction \widehat{E}$, so E would be in NS^T, which is a contradiction. $\qquad$ $_{Claim}\square$ $\qquad\qquad\qquad\square$

Corollary 3.5. *Let T be a nonspecial ω_1-tree, then $\diamondsuit_T \implies \diamondsuit$.*

As mentioned earlier, Kunen proved that $\diamondsuit^-$ and $\diamondsuit$ are equivalent, which in particular allows one to deduce $\diamondsuit$ from $\diamondsuit^*$. We now establish the analogous theorem for the tree-version of these principles. To that end, we first prove a preliminary lemma.

Lemma 3.6. *Let T be a well-pruned ω_1-tree. Then there exists a surjection $f : T \to \omega \times T$ such that for every $t \in T$ of limit height, the restriction $f \upharpoonright t{\downarrow}$ is a bijection onto $\omega \times t{\downarrow}$.*

Proof. Let $C = \mathrm{LIM}(\omega_1) \cup \{0\}$. We will construct a sequence $\langle f_\alpha \mid \alpha \in C \rangle$ such that for every $\alpha \in C$:

(1) $f_\alpha : T_{<\alpha} \to \omega \times T_{<\alpha}$ is a surjective function.

(2) If $\beta < \alpha$ and $\beta \in C$, then $f_\beta \subseteq f_\alpha$.

(3) The restriction $f_\alpha \upharpoonright t{\downarrow}$ is a bijection onto $\omega \times t{\downarrow}$ for every $t \in T_\alpha$.

Observe that if such a sequence $\langle f_\alpha \mid \alpha \in C \rangle$ exists, then $f = \bigcup_{\alpha \in C} f_\alpha$ is a surjection from T onto $\omega \times T$ satisfying the requirements of the lemma.

We proceed by recursion. Let $f_0 = \emptyset$. Now, let $\beta \in C$ and suppose that f_α has been defined for every $\alpha \in C \cap \beta$.

Case $\beta = \alpha + \omega$ for some $\alpha \in C$. Enumerate the level $T_\beta = \{t_n^\beta \mid n \in \omega\}$. We construct a sequence of functions $\langle g_n^\beta \mid n \in \omega \rangle$ such that for every $n \in \omega$:

(i) $\mathrm{dom}(g_n^\beta) = (\bigcup_{i \le n} t_i^\beta {\downarrow}) \setminus T_{<\alpha}$ and the range of g_n^β is contained in $\omega \times T_{<\beta}$.

(ii) $\mathrm{ht}(\pi_1(g_n^\beta(s))) \le \mathrm{ht}(s)$ for every $s \in \mathrm{dom}(g_n^\beta)$.

(iii) $g_n^\beta \subseteq g_{n+1}^\beta$.

(iv) The restriction $g_n^\beta \upharpoonright (t_i^\beta {\downarrow} \setminus T_{<\alpha})$ is a bijection onto $\omega \times (t_i^\beta {\downarrow} \setminus T_{<\alpha})$ for all $i \le n$.

For the base step, let g_0^β be any bijection from $t_0^\beta {\downarrow} \setminus T_{<\alpha}$ onto $\omega \times (t_0^\beta {\downarrow} \setminus T_{<\alpha})$ satisfying condition (ii).

Suppose g_n^β has been defined and let

$$X = \left(t_{n+1}^\beta {\downarrow} \setminus \bigcup_{i \le n} t_i^\beta {\downarrow} \right) \setminus T_{<\alpha} \quad \text{and} \quad Y = \left(\bigcup_{i \le n} (t_i^\beta {\downarrow} \cap t_{n+1}^\beta {\downarrow}) \right) \setminus T_{<\alpha}.$$

Observe that $X \cup Y = t_{n+1}^\beta {\downarrow} \setminus T_{<\alpha}$ and $s < t$ for all $s \in Y$ and $t \in X$.

We want to define g_{n+1}^β. If $t_{n+1}^\beta {\downarrow} = t_i^\beta {\downarrow}$ for some $i \le n$, there is nothing to do; we simply set $g_{n+1}^\beta = g_n^\beta$ and we are done. We may therefore assume that $t_{n+1}^\beta {\downarrow} \ne t_i^\beta {\downarrow}$ for every $i \le n$, which implies, in particular, that X is infinite and Y is finite[7].

Let $R = (\omega \times X) \cup ((\omega \times Y) \setminus g_n^\beta[Y])$ and fix a bijection $h : X \to R$ such that $\mathrm{ht}(\pi_1(h(s))) \le \mathrm{ht}(s)$ for every $s \in X$. Note that $h[X] = R \subseteq \omega \times (t_{n+1}^\beta {\downarrow} \setminus T_{<\alpha})$.

Define $g_{n+1}^\beta = g_n^\beta \cup h$. We claim that g_{n+1}^β satisfies all four conditions. Conditions (i) and (iii) are clear by definition. Condition (ii) holds for g_{n+1}^β because it holds for both g_n^β and h. It remains to verify condition (iv) for $i = n + 1$ (for $i \le n$, it follows from the inductive hypothesis on g_n^β). Note that $g_{n+1}^\beta \upharpoonright (t_{n+1}^\beta {\downarrow} \setminus T_{<\alpha}) = (g_n^\beta \upharpoonright Y) \cup h$.

Claim 3.7. $g_{n+1}^\beta[t_{n+1}^\beta {\downarrow} \setminus T_{<\alpha}] = \omega \times (t_{n+1}^\beta {\downarrow} \setminus T_{<\alpha})$.

Proof of the claim: [$\subseteq$] Let $s \in t_{n+1}^\beta {\downarrow} \setminus T_{<\alpha}$. If $s \in X$, then by definition $g_{n+1}^\beta(s) = h(s) \in R \subseteq \omega \times (t_{n+1}^\beta {\downarrow} \setminus T_{<\alpha})$. If $s \notin X$, then $s \in Y$, so there exists $i \le n$ such that $s \in t_i^\beta {\downarrow}$. Thus $g_{n+1}^\beta(s) = g_n^\beta(s) = (m, t)$. By the inductive hypothesis (conditions (i) and (ii) for g_n^β), we have $\alpha \le \mathrm{ht}(t) \le \mathrm{ht}(s)$.

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[7]Note that if T is Hausdorff, this second case always holds.

By condition (iv), $t \in t_i^\beta{\downarrow} \setminus T_{<\alpha}$. This implies $t \leq s < t_i^\beta$. Since $s < t_{n+1}^\beta$, we have $t < t_{n+1}^\beta$. Given $\mathrm{ht}(t) \geq \alpha$, we conclude $(m, t) \in \omega \times (t_{n+1}^\beta{\downarrow} \setminus T_{<\alpha})$.

[$\supseteq$] Let $(m, t) \in \omega \times (t_{n+1}^\beta{\downarrow} \setminus T_{<\alpha})$. If $t \in X$, we are done as $\omega \times X \subseteq R = h[X]$. So suppose $t \in Y$. If $(m, t) \in g_n^\beta[Y]$, we are done as $g_n^\beta[Y] \subseteq g_{n+1}^\beta[t_{n+1}^\beta{\downarrow} \setminus T_{<\alpha}]$. Otherwise, $(m, t) \in (\omega \times Y) \setminus g_n^\beta[Y] \subseteq R$, so $(m, t) \in h[X]$.

Claim 3.7 implies that $g_{n+1}^\beta \upharpoonright (t_{n+1}^\beta{\downarrow} \setminus T_{<\alpha})$ is surjective onto $\omega \times (t_{n+1}^\beta{\downarrow} \setminus T_{<\alpha})$. To finish, we need to prove that $g_{n+1}^\beta \upharpoonright (t_{n+1}^\beta{\downarrow} \setminus T_{<\alpha})$ is injective.

Since $g_{n+1}^\beta \upharpoonright (t_{n+1}^\beta{\downarrow} \setminus T_{<\alpha}) = (g_n^\beta \upharpoonright Y) \cup h$, and h is injective by construction, we first verify that $g_n^\beta \upharpoonright Y$ is injective. For this, note that there exists $i_0 \leq n$ such that $Y \subseteq t_{i_0}^\beta{\downarrow} \setminus T_{<\alpha}$ (simply take i_0 such that $t_{i_0}^\beta{\downarrow}$ contains the maximal node of the finite chain Y). By the inductive hypothesis (condition (iv)), g_n^β is a bijection on $t_{i_0}^\beta{\downarrow} \setminus T_{<\alpha}$, and thus its restriction to the subset Y is injective.

Since both pieces ($g_n^\beta \upharpoonright Y$ and h) are injective, it suffices to show that their ranges are disjoint, i.e., $h[X] \cap g_n^\beta[Y] = \emptyset$.

First, observe that $g_n^\beta[Y] \cap (\omega \times X) = \emptyset$. To see this, let $s \in Y$ and $s' \in X$. Since $s < s'$, we have $\mathrm{ht}(s) < \mathrm{ht}(s')$. By condition (ii), if we write $g_n^\beta(s) = (m, t)$, then $\mathrm{ht}(t) \leq \mathrm{ht}(s) < \mathrm{ht}(s')$, which implies $t \neq s'$. As this holds for any $s' \in X$, we conclude that $t \notin X$, and therefore $g_n^\beta(s) \notin \omega \times X$. Second, observe that $g_n^\beta[Y]$ is disjoint from $(\omega \times Y) \setminus g_n^\beta[Y]$ simply by definition. Therefore, $h[X] \cap g_n^\beta[Y] = \emptyset$, as desired.

Finally, let $f_\beta = f_\alpha \cup (\bigcup_{n \in \omega} g_n^\beta)$. Condition (iv) implies that f_β satisfies condition (3). Condition (1) follows from the fact that T is well-pruned: for every $s \in T_{<\beta} \setminus T_{<\alpha}$, there exists $n \in \omega$ such that $s \in t_n^\beta{\downarrow}$, which implies that $s \in \mathrm{dom}(f_\beta)$. Furthermore, condition (iv) ensures that for every such s and every $m \in \omega$, the pair (m, s) is in the range of f_β. Finally, condition (2) is satisfied by construction.

Case β is a limit point of $C \cap \beta$. In this case, we simply define $f_\beta = \bigcup_{\alpha \in C \cap \beta} f_\alpha$. $\qquad\qquad\square$

Definition 3.8. If T is a nonspecial tree of height ω_1. A sequence $(\mathcal{D}_t \mid t \in T)$ is called a $\diamondsuit_T^-$-sequence if:

(1) $\mathcal{D}_t \subseteq \mathcal{P}(t{\downarrow})$ and $|\mathcal{D}_t| \leq \omega$ for every $t \in T$.
(2) For every $X \subseteq T$ we have $\{t \in T \mid X \cap t{\downarrow} \in \mathcal{D}_t\} \notin NS^T$.

Theorem 3.9. *Let T be a nonspecial, well-pruned ω_1-tree. Then $\diamondsuit_T^-$ and $\diamondsuit_T$ are equivalent.*

Proof. Let $(\mathcal{D}_t)_{t \in T}$ be a $\diamondsuit_T^-$-sequence and let us fix a surjection $f : T \to \omega \times T$ as in Lemma 3.6 and call $C = \mathrm{LIM}(\omega_1) \cup \{0\}$. Now for every $t \in T$ let $\mathcal{B}_t = \{f[A] \mid A \in \mathcal{D}_t\}$ in case that $t \in T \upharpoonright C$ and $\mathcal{B}_t = \emptyset$ otherwise.

Claim 3.10. For every $X \subseteq \omega \times T$ we have that $\{t \in T \mid X \cap (\omega \times t{\downarrow}) \in \mathcal{B}_t\} \notin NS^T$.

Proof of the Claim. We know that $R := \{t \in T \mid f^{-1}(X) \cap t{\downarrow} \in \mathcal{D}_t\} \notin NS^T$ as $(\mathcal{D}_t)_{t \in T}$ is a $\diamondsuit_T^-$-sequence. On the other hand, by Theorem 2.13, $T \upharpoonright C \in (NS^T)^*$, thus $R \cap (T \upharpoonright C) \notin NS^T$. Now if $t \in R \cap (T \upharpoonright C)$ we know that:

$$f^{-1}(X) \cap t{\downarrow} \in \mathcal{D}_t \implies X \cap f[t{\downarrow}] \in \mathcal{B}_t \implies X \cap (\omega \times t{\downarrow}) \in \mathcal{B}_t,$$

that is, $R \cap (T \upharpoonright C) \subseteq \{t \in T \mid X \cap (\omega \times t\downarrow) \in \mathcal{B}_t\}$, so this last is not in NS^T. $_{Claim}\square$

As each $\mathcal{B}_t$ is countable for every $t \in T \upharpoonright C$, let us write it as $\mathcal{B}_t = \{B_t^k \mid k \in \omega\}$. Clearly in this case $B_t^k \subseteq \omega \times t\downarrow$. Now, let $B_{t,n}^k = \{s \in T \mid (n, s) \in B_t^k\}$ for every $n \in \omega$.

Claim 3.11. There is $n \in \omega$ such that $(B_{t,n}^n)_{t \in T}$ is a $\diamondsuit_T$-sequence.

Proof of the Claim. Suppose that it is false, then for every $n \in \omega$ there is $Q_n \subseteq T$ such that $Z_n := \{t \in T \mid Q_n \cap t\downarrow = B_{t,n}^n\} \in NS^T$. Now, let $Q := \bigcup_{n \in \omega}(\{n\} \times Q_n)$. For every $n \in \omega$ we know that $Y_n := \{t \in T \mid Q \cap (\omega \times t\downarrow) = B_t^n\} \in NS^T$. Indeed, to see that $Y_n \in NS^T$ it is enough to see that $Y_n \subseteq Z_n$, so:

$$t \in Y_n \implies Q \cap (\omega \times t\downarrow) = B_t^n \implies \{s \in t\downarrow \mid (n, s) \in Q \cap (\omega \times t\downarrow)\} = B_{t,n}^n \implies$$

$$\{s \in t\downarrow \mid (s \in t\downarrow) \wedge (s \in Q_n)\} = B_{t,n}^n \implies Q_n \cap t\downarrow = B_{t,n}^n \implies t \in Z_n.$$

Now, since the ideal NS^T is ω_1-complete (part (II) of Theorem 2.12), we know that the set $\bigcup_{n \in \omega} Y_n = \{t \in T \mid \exists n \in \omega(B \cap (\omega \times t\downarrow) = B_t^n)\} \in NS^T$, which is a contradiction to Claim 3.10.

$_{Claim}\square$ $\square$

As we said before, a direct consequence of Theorem 3.9 is that $\diamondsuit_T^* \implies \diamondsuit_T$ for every nonspecial well-pruned ω_1-tree T.

What we consider the most natural next step is to explore how $\diamondsuit_S$ and $\diamondsuit_T$ compare when S is a nonspecial subtree of T. For this, it is convenient to recall the natural ordering on arbitrary trees:

Notation 3.12. If S and T are trees, we write $S \leq T$ to mean that there exists a strictly increasing map $g \colon S \to T$, i.e., such that $g(s) <_T g(t)$ whenever $s <_S t$.

Observe that if $g \colon S \to T$ is strictly increasing, then the map $f(s) = g(s) \upharpoonright \mathrm{ht}(s)$ is also strictly increasing. As defined in [30], a strictly increasing and level-preserving[8] map $f \colon S \to T$ is called a *Lipschitz map*[9]. Hence, whenever $S \leq T$ and $f \colon S \to T$ witnesses this relation, we will assume that f is Lipschitz.

However, a Lipschitz map $f \colon S \to T$ need not be injective: it may send distinct nodes of the same level in S to a single node in T. For instance, for any tree T of height ω_1 we have $T \leq \omega_1$ via the map sending each node to its height. Conversely, T has a cofinal branch precisely when $\omega_1 \leq T$.

Note that if $f \colon S \to T$ is Lipschitz and $A \subseteq T$ is an antichain, then $f^{-1}(A)$ is also an antichain. Indeed, if $s <_S t$ both lie in $f^{-1}(A)$, then $f(s) <_T f(t)$ and both are in A, contradicting that A is an antichain. As a consequence, the preimage of any special subset $B \subseteq T$ is special; in particular we have the following fact.

Fact 3.13. If $S \leq T$ and S is nonspecial, then T is nonspecial.

Notation 3.14. Let $s \in S$ and $\alpha \leq \mathrm{ht}(s)$. We denote by $s \upharpoonright \alpha$ the unique predecessor of s with height α, that is, the unique $u \in S$ with $u <_S s$ and $\mathrm{ht}(u) = \alpha$.

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[8]A map $f \colon S \to T$ is *level-preserving* if $\mathrm{ht}(s) = \mathrm{ht}(f(s))$ for every $s \in S$.

[9]Todorcevic's notion is more general as it allows partial maps, but when the domain is downward closed (in particular, when it is all of S) it is equivalent to being strictly increasing and level-preserving.

Fact 3.15. Let $f \colon S \to T$ be a Lipschitz map and let $s \in S$. Then for every $\alpha \leq \mathrm{ht}(s)$,

$$f\big(s \restriction \alpha\big) \;=\; f(s) \restriction \alpha.$$

Proof. Set $v = s \restriction \alpha$. Note that $f(v)$ satisfies:

- $f(v) <_T f(s)$ since $v <_S s$, and
- $\mathrm{ht}(f(v)) = \alpha = \mathrm{ht}(f(s) \restriction \alpha)$.

Since $f(s) \restriction \alpha$ is the unique node in T with these properties, we necessarily have $f(v) = f(s) \restriction \alpha$. $\qquad\square$

In order to state the following result, we first need to introduce the Katětov order.

Definition 3.16. [19] Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on X and Y respectively and $f \colon Y \to X$.

(1) f is a *Katětov* function from $(Y, \mathcal{J})$ to $(X, \mathcal{I})$ if $f^{-1}(A) \in \mathcal{J}$ for all $A \in \mathcal{I}$.

(2) $\mathcal{I} \leq_K \mathcal{J}$ ($\mathcal{I}$ *is Katětov below* $\mathcal{J}$) if there exists a Katětov function from $(Y, \mathcal{J})$ to $(X, \mathcal{I})$.

The interested reader is referred to [12, 13] for further details on the Katětov order.

Theorem 3.17. *Let S and T be trees of height ω_1 with $S \leq T$. Then:*

(i) $NS^T \leq_K NS^S$, and

(ii) If S is a nonspecial ω_1-tree[10], then $\Diamond_S$ implies $\Diamond_T$.

Proof. Let $f \colon S \to T$ be a Lipschitz map. To prove (i), we show that f is a Katětov function from (T, NS^T) to (S, NS^S), i.e., whenever $X \subseteq T$ satisfies $X \in NS^T$, then $f^{-1}(X) \in NS^S$. Fix such an $X \in NS^T$. By definition,

$$X = \bigcup_{t \in T} \big(A_t \cap t{\uparrow}\big),$$

where each A_t is special, and hence each $f^{-1}(A_t)$ is also special.

For each $s \in S$, set $B_s = f^{-1}\big(A_{f(s)}\big)$, which is special. We will show

$$f^{-1}(X) \;\subseteq\; \bigcup_{s \in S} \big(B_s \cap s{\uparrow}\big),$$

and since $\bigcup_{s \in S}(B_s \cap s{\uparrow}) \in NS^S$, this will prove $f^{-1}(X) \in NS^S$.

Take any $u \in f^{-1}(X)$. Then $f(u) \in X = \bigcup_{t \in T}(A_t \cap t{\uparrow})$, so pick $t \in T$ with $f(u) \in A_t \cap t{\uparrow}$. Let $\alpha = \mathrm{ht}(t)$ and $s = u \restriction \alpha$. By Proposition 3.15, $f(s) = t$. Now

$$u \in B_s \iff u \in f^{-1}(A_{f(s)}) \iff u \in f^{-1}(A_t) \iff f(u) \in A_t,$$

where the final condition holds by choice of t. Clearly $u \in s{\uparrow}$, and hence $u \in B_s \cap s{\uparrow}$, as required.

To prove (ii) assume S is an ω_1-tree and let $(D_s)_{s \in S}$ be a $\Diamond_S$-sequence. For each $t \in T$, set

$$\mathcal{D}_t = \{\, f[D_s] \mid s \in S, \; f(s) = t\}.$$

Note that each $\mathcal{D}_t$ is a subfamily of $\mathcal{P}(t{\downarrow})$ and is countable, since $f^{-1}(\{t\}) \subseteq S_{\mathrm{ht}(t)}$ and $S_{\mathrm{ht}(t)}$ is countable because S is an ω_1-tree.

We will show that $(\mathcal{D}_t)_{t \in T}$ is a $\Diamond_T^{-}$-sequence. Fix $X \subseteq T$.

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[10]By Fact 3.13, this implies that T is also nonspecial, in particular it makes sense to consider $\Diamond_T$.

Claim 3.18. For any $s \in S$, we have $f\big(s{\downarrow} \cap f^{-1}(X)\big) = f(s){\downarrow} \cap X$.

Proof of Claim: "$\subseteq$": If $u \in s{\downarrow} \cap f^{-1}(X)$ then $u <_S s$ and $f(u) \in X$. Since f is a Lipschitz map, $f(u) <_T f(s)$, so $f(u) \in f(s){\downarrow} \cap X$.

"$\supseteq$": Let $w \in f(s){\downarrow} \cap X$ and call $v = s \upharpoonright \mathrm{ht}(w)$. By Fact 3.15, we have that $f(v) = f(s) \upharpoonright \mathrm{ht}(w) = w$. Since $w \in X$, it follows that $v \in f^{-1}(X)$, and hence $v \in s{\downarrow} \cap f^{-1}(X)$, proving $w \in f\big(s{\downarrow} \cap f^{-1}(X)\big)$.
$_{Claim}\square$

Let

$$Y = \{\, t \in T \mid X \cap t{\downarrow} \in \mathcal{D}_t \}.$$

We must show $Y \notin NS^T$. Suppose otherwise. Then by (i), $f^{-1}(Y) \in NS^S$. On the other hand, since (D_s) is a $\Diamond_S$-sequence,

$$Z = \{\, s \in S \mid f^{-1}(X) \cap s{\downarrow} = D_s \} \notin NS^S.$$

But Claim 3.18 implies that

$$s \in Z \implies f(s) \in Y,$$

so $Z \subseteq f^{-1}(Y)$, contradicting the nonstationarity of $f^{-1}(Y)$. Hence $Y \notin NS^T$, and $(\mathcal{D}_t)_{t \in T}$ is indeed a $\Diamond_T^-$-sequence. Finally, by Theorem 3.9, we conclude that $\Diamond_T$ holds. $\square$

Note that part (ii) of Theorem 3.17 provides an alternative proof of Corollary 3.5, since if T is a nonspecial ω_1-tree, then $T \leq \omega_1$. Another consequence of Theorem 3.17 is the following.

Corollary 3.19. *Let S and T be nonspecial trees of height ω_1 such that S is a subtree of T. Then $\Diamond_S$ implies $\Diamond_T$.*

Proof. The inclusion $S \hookrightarrow T$ is a Lipschitz map, so part (ii) of Theorem 3.17 gives $\Diamond_S \implies \Diamond_T$. $\square$

A particularly interesting case is when S is a cofinal branch of T. Note that in this scenario, we have $\omega_1 \leq S \leq \omega_1$, which implies that S is a nonspecial ω_1-tree and, by part (ii) of Theorem 3.17, $\Diamond_S$ and $\Diamond$ are equivalent. In particular, by Corollary 3.19, we have that $\Diamond$ implies $\Diamond_T$. This, combined with Corollary 3.5, yields the following result.

Corollary 3.20. *For a tree T of height ω_1 with a cofinal branch, $\Diamond$ implies $\Diamond_T$. Additionally, if T is an ω_1-tree, then $\Diamond$ and $\Diamond_T$ are equivalent.*

An important nonspecial ω_1-tree is

$$T(\emptyset) = \{s \in 2^{<\omega_1} \mid |\operatorname{supp}(s)| < \omega\},$$

which possesses several interesting properties. In particular, it is shown in [33, Theorem 4.1 and Corollary 4.4] that $T(\emptyset)$ has no Aronszajn subtrees and is *almost-Suslin* (Definition 4.10). A natural question is what can be said about $\Diamond_{T(\emptyset)}$. Corollary 3.20 answers it: since $T(\emptyset)$ has a cofinal branch, $\Diamond_{T(\emptyset)}$ is equivalent to the usual $\Diamond$; indeed, as we will show in Theorem 4.13, being almost-Suslin is a sufficient condition for this equivalence.

We now turn to a different topic. One of the earliest results, if not the very first, when studying $\Diamond$ is that it implies the Continuum Hypothesis (CH). Therefore, when studying $\Diamond_T$, it is natural

to ask what cardinal arithmetic constraints $\Diamond_T$ imposes. Of course, by Corollary 3.5, we know that $\Diamond_T$ implies CH whenever T is an ω_1-tree. However, it is interesting to explore whether $\Diamond_T$ generally provides a cardinal inequality associated with the levels of the tree T.

Recall that the standard proof that $\Diamond$ implies CH shows that if $(D_\alpha)_{\alpha<\omega_1}$ is a $\Diamond$-sequence, then for every subset $X \subseteq \omega$, there exists some $\alpha \in \omega_1 \setminus \omega$ with $X = D_\alpha$. Consequently,

$$|\mathcal{P}(\omega)| \leq |\omega_1 \setminus \omega| = \omega_1,$$

so CH follows.

In other words, when we think of ω_1 as a tree T, the above can be reformulated as $\Diamond_T$ implying that $|\mathcal{P}(T_{<\omega})| \leq |T_{\geq\omega}|$. It is natural to ask whether this inequality holds for any tree T of height ω_1 whenever $\Diamond_T$ holds. However, this is not true in general. For example, consider a tree S of height ω with a cofinal branch $\{r_n \mid n \in \omega\}$ and such that $|S| = 2^{\omega_1}$. Now, let $R = \{r_\alpha \mid \alpha \in \omega_1 \setminus \omega\}$ be a set disjoint from S, where $r_\alpha \neq r_\beta$ if $\alpha \neq \beta$. Define $T = S \cup R$ as the tree where $r_\alpha\!\downarrow = \{r_\beta \mid \beta < \alpha\}$ and that has S a subtree. Suppose $\Diamond$ holds; since T has a cofinal branch, by Corollary 3.20, $\Diamond_T$ also holds. However, note that

$$|T_{\geq\omega}| = \omega_1 < |T| = 2^{\omega_1} < |\mathcal{P}(T_{<\omega})| = |\mathcal{P}(S)| = |\mathcal{P}(2^{\omega_1})| = 2^{2^{\omega_1}}.$$

This yields the following result:

Proposition 3.21. *It is consistent that there exists a nonspecial tree T of height ω_1 such that $\Diamond_T$ holds and $|T_{\geq\omega}| < |T| < |\mathcal{P}(T_{<\omega})|$.*

Although $|\mathcal{P}(T_{<\omega})| \leq |T_{\geq\omega}|$ does not necessarily hold whenever $\Diamond_T$ holds, we can still prove the following:

Proposition 3.22. *Let T be a nonspecial tree of height ω_1. Then $\Diamond_T$ implies $2^\omega \leq |T_{\geq\omega}|$.*

Proof. Let $(D_t)_{t\in T}$ be a $\Diamond_T$-sequence. By Theorem 2.13, $T_{\geq\omega} = T \upharpoonright (\omega_1 \setminus \omega) \in (NS^T)^*$. Now fix $A \subseteq \omega$. Since $(D_t)_{t\in T}$ is a $\Diamond_T$-sequence, there exists $t_A \in T_{\geq\omega}$ such that $(T \upharpoonright A) \cap t_A\!\downarrow = D_{t_A}$, i.e., $\{s < t_A \mid \mathrm{ht}(s) \in A\} = D_{t_A}$. It is straightforward to verify that $t_A \neq t_B$ if $A \neq B$. Therefore, $2^\omega \leq |T_{\geq\omega}|$. $\qquad\square$

In particular, as a consequence of Proposition 3.22, we have an alternative proof of the fact that if T is a nonspecial ω_1-tree, then $\Diamond_T$ implies CH, since in that case $|T_{\geq\omega}| = \omega_1$.

Given the flexibility of the definition of $\Diamond_T$, which allows us to consider any tree of height ω_1, a natural question arises: does there exist a tree T such that $\Diamond_T$ is provable in ZFC? We know that for $T = \omega_1$ the answer is negative, since $\Diamond$ is independent of ZFC.

However, as we will see, $2^{<\omega_1}$ is a tree whose diamond principle holds in ZFC. To prepare for that argument, it is convenient to fix some notation.

Notation 3.23. If T is a tree and $t \in T$, then $\mathrm{Succ}_T(t)$ denotes the set

$$\{\, s \in T \mid t <_T s \text{ and } \mathrm{ht}(s) = \mathrm{ht}(t) + 1 \,\},$$

that is, the collection of immediate successors of t in T.

Definition 3.24. Let T be a tree.

(1) A family $\mathcal{Q} = \{Q_t \mid t \in T\}$ is a *successor-partition of* T if for every $t \in T$ the set $Q_t = \{Q_t(0), Q_t(1)\}$ is a partition of $\mathrm{Succ}_T(t)$; i.e., $Q_t(0) \cup Q_t(1) = \mathrm{Succ}_T(t)$ and $Q_t(0) \cap Q_t(1) = \emptyset$.

(2) Given a successor-partition $\mathcal{Q}$ of T and a function $f \colon T \to 2$, the subtree of T determined by $\mathcal{Q}$ and f, denoted $T(\mathcal{Q}, f)$, is the unique subtree of T satisfying:

  (a) If $t \in T(\mathcal{Q}, f)$, then $\mathrm{Succ}_{T(\mathcal{Q},f)}(t) = Q_t(f(t))$.

  (b) If $t \in T$, $\mathrm{ht}(t)$ is a limit ordinal and $t \restriction \beta \in T(\mathcal{Q}, f)$ for every $\beta < \mathrm{ht}(t)$, then $t \in T(\mathcal{Q}, f)$.

(3) A successor-partition $\mathcal{Q}$ of T is *nice* if $T(\mathcal{Q}, f)$ is nonspecial for every $f \colon T \to 2$.

Theorem 3.25. *Let T be a tree of height ω_1. If T admits a nice successor-partition, then $\diamondsuit_T$ holds.*

Proof. For each $t \in T$ define $D_t = \{t \restriction \alpha \mid \alpha < \mathrm{ht}(t) \text{ and } t \restriction (\alpha + 1) \in Q_{t \restriction \alpha}(1)\}$. For any $X \subseteq T$ let $\chi_X \colon T \to 2$ be the characteristic function of X. Since $\mathcal{Q}$ is nice, the subtree $S := T(\mathcal{Q}, \chi_X)$ is nonspecial.

Claim 3.26. $S \subseteq \{t \in T \mid X \cap t{\downarrow} = D_t\}$.

  Proof of the claim. Fix $t \in S$ and $\alpha < \mathrm{ht}(t)$. We show that $t \restriction \alpha \in D_t$ if and only if $t \restriction \alpha \in X$.

  By definition,

(1) $$t \restriction \alpha \in D_t \iff t \restriction (\alpha + 1) \in Q_{t \restriction \alpha}(1)$$

On the other hand, because S is a subtree of T, $t \in S$ and $t \restriction (\alpha + 1) \leq_T t$, we also have $t \restriction (\alpha + 1) \in S$, hence

(2) $$t \restriction (\alpha + 1) \in Q_{t \restriction \alpha}(\chi_X(t \restriction \alpha))$$

Therefore, by (1) and (2), we have

$$t \restriction (\alpha + 1) \in Q_{t \restriction \alpha}(1) \iff \chi_X(t \restriction \alpha) = 1.$$

This yields $t \restriction \alpha \in D_t \iff \chi_X(t \restriction \alpha) = 1$, i.e., $t \restriction \alpha \in D_t \iff t \restriction \alpha \in X$. Consequently $X \cap t{\downarrow} = D_t$, as required. $\hfill{}_{Claim}\square$

  Since S is nonspecial and it is a subtree of T, Corollary 2.18 yields $S \notin NS^T$. Hence

$$\{t \in T \mid X \cap t{\downarrow} = D_t\} \notin NS^T.$$

As X was arbitrary, the sequence $(D_t)_{t \in T}$ is a witness of $\diamondsuit_T$. $\hfill\square$

Lemma 3.27. $2^{<\omega_1}$ *admits a nice successor-partition.*

Proof. Let $T = 2^{<\omega_1}$. For each $t \in T$ define $Q_t(0) = \{t^\frown 0\}$ and $Q_t(1) = \{t^\frown 1\}$ and set $\mathcal{Q} = \{Q_t \mid t \in T\}$. Fix $f \colon T \to 2$ and consider the subtree $T(\mathcal{Q}, f)$. Define a sequence $(s_\alpha)_{\alpha < \omega_1}$ by $s_0 = \emptyset$, $s_{\alpha+1} = s_\alpha^\frown f(s_\alpha)$, and for limit α set $s_\alpha = \bigcup_{\beta < \alpha} s_\beta$. Then $T(\mathcal{Q}, f) = \{s_\alpha \mid \alpha < \omega_1\}$, so $T(\mathcal{Q}, f)$ is a cofinal branch of T, in particular $T(\mathcal{Q}, f)$ is nonspecial. Therefore $\mathcal{Q}$ is nice. $\hfill\square$

  As a consequence of Theorem 3.25 and Lemma 3.27, we have:

Theorem 3.28. $\diamondsuit_{2^{<\omega_1}}$ *holds in* ZFC.

Two canonical examples of nonspecial ω_1-trees without cofinal branches are the tree $\sigma\mathbb{Q}$ (first studied by Kurepa [24, 23]) and the "shooting-a-club" tree $T(S)$ (where S is a bistationary subset of ω_1). The latter was first introduced by Baumgartner, Harrington, and Kleinberg in [1] as a forcing notion and was later studied by Todorcevic as a tree in [35], where several interesting facts were established; for example, $T(S) \times T(S')$ is special if and only if $S \cap S' \in NS_{\omega_1}$, which settles Hedetniemi's Conjecture at that level.

The tree $\sigma\mathbb{Q}$ is the set $\{\, t \mid \exists \alpha < \omega_1 \ (t\colon \alpha \to \mathbb{Q} \text{ increasing and } \mathrm{ran}(t) \text{ is bounded}) \,\}$ ordered by $\subseteq$. A natural successor-partition of $\sigma\mathbb{Q}$ is obtained by fixing a partition $\mathbb{Q} = \mathbb{Q}_0 \cup \mathbb{Q}_1$ in which both $\mathbb{Q}_0$ and $\mathbb{Q}_1$ are dense in $\mathbb{Q}$. For each $t \in \sigma\mathbb{Q}$ and $i \in \{0,1\}$ set $Q_t(i) = \{\, s \in \mathrm{Succ}_{\sigma\mathbb{Q}}(t) \mid s(\mathrm{ht}(t)) \in \mathbb{Q}_i \,\}$. It is natural to ask whether the successor-partition $\mathcal{Q}$ is nice; that is, whether for every $f\colon \sigma\mathbb{Q} \to 2$, the subtree $\sigma\mathbb{Q}(\mathcal{Q}, f)$ is nonspecial. The standard proof that $\sigma\mathbb{Q}$ is nonspecial, as shown in [33, Corollary 9.9], does not seem to immediately adapt to prove $\sigma\mathbb{Q}(\mathcal{Q}, f)$ is nonspecial.

Question 3.29. *Is the successor-partition $\mathcal{Q}$ described above nice? Or more generally: does $\Diamond_{\sigma\mathbb{Q}}$ hold?*

The "shooting-a-club" tree is defined as follows. Fix a bistationary set $S \subseteq \omega_1$. Let $T(S)$ be the collection of bounded closed subsets $p \subseteq S$ of ordinals, ordered by end-extension: $q \leq p$ iff $q = p \cap \alpha$ for some $\alpha < \omega_1$. It is straightforward to verify that $T(S)$ is well-pruned, has no cofinal branch, and is nonspecial. The last assertion follows from the fact that any special well-pruned tree of height ω_1 collapses ω_1, whereas $T(S)$ does not collapse ω_1; in fact $T(S)$ is ω-distributive (see [1, Theorem B] or [16, Lemma 23.9, p. 445]). As with $\sigma\mathbb{Q}$, it is natural to ask:

Question 3.30. *Given a bistationary $S \subseteq \omega_1$, does $\Diamond_{T(S)}$ hold?*

Finally, we conclude the section by showing that $\Diamond_T$ can be expressed within a more familiar framework, a connection that Assaf Rinot brought to our attention, and for which we express our gratitude.

Definition 3.31. Let $\mathcal{I}$ be an ideal over the powerset of some set Z, i.e., $\mathcal{I} \subseteq \mathcal{P}(\mathcal{P}(Z))$. We say that $\Diamond(\mathcal{I})$ holds if there exists a function $g : \mathcal{P}(Z) \to \mathcal{P}(Z)$ such that for every $X \subseteq Z$, the collection $\{Y \subseteq Z \mid X \cap Y = g(Y)\}$ is in $\mathcal{I}^+$.

Now suppose that $\Diamond(\mathcal{I})$ holds and let $g : \mathcal{P}(Z) \to \mathcal{P}(Z)$ be a witness to this. Now consider $f : \mathcal{P}(Z) \to \mathcal{P}(Z)$ defined by $f(Y) = g(Y) \cap Y$. Then f is also a witness of $\Diamond(\mathcal{I})$. Indeed, it suffices to note that for every $X \subseteq Z$, we have:

$$\{Y \subseteq Z \mid X \cap Y = g(Y)\} = \{Y \subseteq Z \mid X \cap Y = g(Y) \cap Y\} = \{Y \subseteq Z \mid X \cap Y = f(Y)\}.$$

Thus, without loss of generality, we can assume that $g(Y) \subseteq Y$ for all $Y \subseteq Z$.

The language of Definition 3.31 is usually used to define the diamond principle on $P_\kappa(\lambda)$ (see, for example, [27]), but it also captures $\Diamond_T$, as we will see.

Let T be a tree of height ω_1, and let $\mathcal{J}_T \subseteq \mathcal{P}(\mathcal{P}(T))$ be the ideal defined as follows:

(1) $\{Y \subseteq T \mid \forall t \in T (t\!\downarrow \neq Y)\} \in \mathcal{J}_T$, i.e., $\{t\!\downarrow \mid t \in T\} \in (\mathcal{J}_T)^*$.

(2) If $A \subseteq \{t{\downarrow} \mid t \in T\}$, then $A \in \mathcal{J}_T$ if and only if $\{t \in T \mid t{\downarrow} \in A\} \in NS^T$.

Essentially, $\mathcal{J}_T$ is designed to mimic the ideal NS^T but over $\mathcal{P}(T)$.

Proposition 3.32. $\Diamond_T$ *is equivalent to* $\Diamond(\mathcal{J}_T)$.

Proof. Suppose that $(D_t)_{t \in T}$ is a $\Diamond_T$-sequence, and let $g : \mathcal{P}(T) \to \mathcal{P}(T)$ be defined by $g(t{\downarrow}) = D_t$ for all $t \in T$, and constant $\emptyset$ for the rest of $\mathcal{P}(T)$. It is clear that for every $X \subseteq T$, we have:

$$\{Y \subseteq T \mid X \cap Y = g(Y)\} \supseteq \{t{\downarrow} \mid X \cap t{\downarrow} = D_t\}$$

and the latter set is in $\mathcal{J}_T^+$ since $\{t \mid X \cap t{\downarrow} = D_t\} \notin NS^T$, as $(D_t)_{t \in T}$ is a $\Diamond_T$-sequence.

Conversely, suppose that g is a witness of $\Diamond(\mathcal{J}_T)$ such that $g(Y) \subseteq Y$ for every $Y \subseteq T$. Now, for every $t \in T$, let $D_t = g(t{\downarrow})$. Note that since $\{Y \subseteq T \mid \forall t \in T(t{\downarrow} \neq Y)\} \in \mathcal{J}_T$, for every X we have $\{t{\downarrow} \mid X \cap t{\downarrow} = D_t\} \notin \mathcal{J}_T$. Therefore, $\{t \mid X \cap t{\downarrow} = D_t\} \notin NS^T$, which proves that $(D_t)_{t \in T}$ is a $\Diamond_T$-sequence. $\qquad\square$

4. THE CONSISTENCY OF $\Diamond_T$ AND THE CASE FOR ALMOST-SUSLIN TREES

Although we already know some results regarding $\Diamond_T$, we still lack a crucial piece of information: its consistency. The first thing we will do in this section is to establish that consistency for some class of trees.

Theorem 4.1. *Let T be a nonspecial ω_1-tree. Then $\Diamond^*$ implies $\Diamond_T^*$.*

Proof. Let $(\mathcal{A}_\alpha \mid \alpha \in \omega_1)$ be a $\Diamond^*$-sequence and enumerate $\mathcal{A}_\alpha = \{A_\alpha^n \mid n \in \omega\}$ for every $\alpha \in \omega_1$. Now assume without loss of generality that $(T, <_T) = (\omega_1, <_T)$. We know that the set

$$C = \{\alpha \in \omega_1 \mid T_{<\alpha} = \alpha\}$$

is club, consequently, by condition (iv) of Lemma 2.13, $T \restriction C \notin NS^T$. Now for every $t \in T \restriction C$ let $\mathcal{D}_t = \{A_\alpha^n \cap t{\downarrow} \mid n \in \omega\}$ and if $t \notin T \restriction C$ let $\mathcal{D}_t = \emptyset$.

Claim 4.2. $(\mathcal{D}_t \mid t \in T)$ is a $\Diamond_T^*$-sequence.

Proof of the Claim. Let $X \subseteq T$. As $T = \omega_1$, $D = \{\alpha \in \omega_1 \mid X \cap \alpha \in \mathcal{A}_\alpha\}$ contains a club, hence so does $E = D \cap C$. Let any $t \in T \restriction E$ and let $\alpha = \mathrm{ht}(t)$. Now we know that $X \cap T_{<\alpha} = X \cap \alpha \in \mathcal{A}_\alpha$, so $X \cap \alpha = A_\alpha^n$ for some $n \in \omega$, thus:

$$X \cap t{\downarrow} = (X \cap \alpha) \cap t{\downarrow} = A_\alpha^n \cap t{\downarrow} \in \mathcal{D}_t.$$

This proves that $\{t \in T \mid X \cap t{\downarrow} \notin \mathcal{D}_T\} \subseteq T \restriction (\omega_1 \setminus E) = T \setminus (T \restriction E) \in NS^T$ as $(T \restriction E) \in (NS^T)^*$ by condition (v) of Lemma 2.13. $\qquad\qquad _{Claim}\square \qquad\qquad\qquad\qquad \square$

It is natural to ask whether the proof of Theorem 4.1 can be adapted to show that $\Diamond$ implies $\Diamond_T$. However, note that this seems unlikely because in the proof we use the fact that if $D \subseteq \omega_1$ (the set where X is guessed) is a club, then two things happen:

(i) $E = D \cap C$ is club, and

(ii) $T \restriction E \in (NS^T)^*$.

Note that if D, the set where X is guessed, is merely stationary (not necessarily club), although condition (i) has an adequate analogue (as $E = D \cap C$ is stationary), the analogue of condition (ii) fails, since we cannot guarantee that $T \restriction E \notin NS^T$. As we will see later, this is actually the only obstruction to deducing $\diamondsuit_T$ from $\diamondsuit$ (Lemma 4.8). However, since in principle not every well-pruned ω_1-tree satisfies this property (as Theorem 6.4 witness), in order to prove that $\diamondsuit_T$ is consistent for every well-pruned nonspecial ω_1-tree, we must go further and obtain $\diamondsuit_T$ not as a consequence of $\diamondsuit$ but from $\diamondsuit^*$:

Corollary 4.3. *Let T be a well-pruned nonspecial ω_1-tree. Then $\diamondsuit^*$ implies $\diamondsuit_T$.*

Proof. $\diamondsuit^*$ implies $\diamondsuit^*_T$ by Theorem 4.1, while $\diamondsuit^*_T$ implies $\diamondsuit_T$ by Theorem 3.9. □

In particular:

Corollary 4.4. $L \models \forall T$ *nonspecial well-pruned ω_1-tree, $\diamondsuit_T$ holds.*

It would be interesting to know whether a direct proof of Corollary 4.4 can be given using the fine structure of L or the Condensation Lemma.

We now aim to prove that for every well-pruned, nonspecial Aronszajn tree T, $\diamondsuit_T$ is consistent. By Corollary 4.3, this reduces to showing that one can force $\diamondsuit^*$ while preserving the fact that T remains a nonspecial Aronszajn tree (being well-pruned is preserved trivially). To this end we need the following lemmas, due to Silver and Todorcevic respectively, which will also be used in subsequent sections.

Lemma 4.5. [28, p. 387] *Let T be a well-pruned Aronszajn tree and let $\mathbb{P}$ be a σ-closed forcing. Then $\mathbb{P} \Vdash$ "T is a well-pruned Aronszajn tree."*

Lemma 4.6. [32, Lemma 12] *Let T be a nonspecial ω_1-tree and let $\mathbb{P}$ be a σ-closed forcing. Then $\mathbb{P}$ does not specialize T.*

We can now prove the desired result:

Theorem 4.7. *Let T be a well-pruned, nonspecial Aronszajn tree. Then in some cofinality-preserving forcing extension, T is a well-pruned nonspecial Aronszajn tree and $\diamondsuit_T$ holds.*

Proof. As previously stated, it suffices to show that we can force $\diamondsuit^*$ with a cofinality-preserving forcing that preserves T as a well-pruned, nonspecial Aronszajn tree. By [21, Exercise H20, p. 249], under CH, there exists a σ-closed forcing $\mathbb{Q}$ such that $\mathbb{Q} \Vdash$ "$\diamondsuit^*$" (this forcing plays a key role in Section 5 and additionally forces a stronger diamond principle, but this property is sufficient for now). Furthermore, it is well-known that $\mathbb{P} = (2^{<\omega_1}, \supseteq)$ is σ-closed and forces CH. Thus, $\mathbb{P} \star \dot{\mathbb{Q}}$ is a σ-closed forcing that forces $\diamondsuit^*$. By Lemmas 4.5 and 4.6, T remains a well-pruned, nonspecial Aronszajn tree in $V[G]$. Hence, $V[G] \models \diamondsuit_T$. □

It is natural to think that for trees that are not too wide, $\diamondsuit_T$ should be very similar to $\diamondsuit$. As we mentioned earlier, the issue with trying to deduce $\diamondsuit_T$ from $\diamondsuit$ is that we do not know that $T \restriction X \notin NS^T$

when $X \subseteq \omega_1$ is stationary. As we will see, that this is actually the only obstruction (Lemma 4.8), and that almost-Suslin trees (Definition 4.10) do not have this obstruction (Lemma 4.11).

Lemma 4.8. *Let T be a nonspecial ω_1-tree such that for every $X \subseteq \omega_1$, $T \upharpoonright X \in NS^T$ if and only if X is nonstationary. Then $\diamondsuit$ and $\diamondsuit_T$ are equivalent.*

Proof. By Corollary 3.5, we know that $\diamondsuit_T$ implies $\diamondsuit$. Now suppose that $(A_\alpha \mid \alpha \in \omega_1)$ is a $\diamondsuit$-sequence and assume without loss of generality that $(T, <_T) = (\omega_1, <_T)$. We know that the set

$$C = \{\alpha \in \omega_1 \mid T_{<\alpha} = \alpha\}$$

is club, consequently, by condition (iv) of Lemma 2.13, $T \upharpoonright C \in (NS^T)^*$. For every $t \in T$ let $D_t \subseteq t{\downarrow}$ be as follows:

$$D_t = \begin{cases} A_\alpha \cap t{\downarrow} & \text{if } t \in T \upharpoonright C \\ \emptyset & \text{if } t \notin T \upharpoonright C. \end{cases}$$

Claim 4.9. $(D_t \mid t \in T)$ is a $\diamondsuit_T$-sequence.

Proof of the Claim. Let $X \subseteq T$. As $T = \omega_1$ then $D = \{\alpha \in \omega_1 \mid X \cap \alpha = A_\alpha\}$ is stationary in ω_1, so it is $E = D \cap C$ and thus $T \upharpoonright E \notin NS^T$. Let any $t \in T \upharpoonright E$ and let $\alpha = \mathrm{ht}(t)$. Now we know that $X \cap T_{<\alpha} = X \cap \alpha = A_\alpha$, thus:

$$X \cap t{\downarrow} = (X \cap \alpha) \cap t{\downarrow} = A_\alpha \cap t{\downarrow} = D_t.$$

This proves that $T \upharpoonright E \subseteq \{t \in T \mid X \cap t{\downarrow} = D_T\}$ so this last set is not in NS^T. $_{Claim}\square$ $\square$

Definition 4.10. [8] Let T be an ω_1-tree. We say that T is *almost-Suslin* if, for every antichain $A \subseteq T$, the set $\widehat{A} = \{\mathrm{ht}(t) \mid t \in A\}$ is nonstationary in ω_1.

Lemma 4.11. *Let T be an almost-Suslin tree. Then, for every $X \subseteq \omega_1$, $T \upharpoonright X \in NS^T$ if and only if X is nonstationary.*

Proof. By condition (ii) of Lemma 2.13, we know that if $X \subseteq \omega_1$ is nonstationary, then $T \upharpoonright X \in NS^T$. Thus, it is enough to prove the converse. Suppose that $T \upharpoonright X \in NS^T$, which means that there is a collection $\{A_t \mid t \in T\}$ of special sets of T such that $T \upharpoonright X = \nabla_{t \in T} A_t$. Hence, for every $t \in T$, A_t is a countable union of antichains, and since T is almost-Suslin, every antichain is such that the set of its heights is nonstationary, meaning that $\widehat{A_t}$ is itself nonstationary.

Now, for every $s \in T \upharpoonright X$, let $t(s) \in T$ be a node of minimal height such that $s \in A_t \cap t{\uparrow}$. For every $\alpha \in X$, let $g(\alpha) = \min\{\mathrm{ht}(t(s)) \mid s \in T_\alpha\}$. Clearly, $g : X \to \omega_1$ is a regressive function. The proof will be complete with the following claim:

Claim 4.12. $g^{-1}(\{\beta\})$ is nonstationary for every $\beta \in \omega_1$.

Proof of the Claim. Suppose that there exist $\beta \in \omega_1$ and a stationary set $Y \subseteq X$ such that $g(\alpha) = \beta$ for all $\alpha \in Y$. This implies that for every $\alpha \in Y$, there exist $s(\alpha) \in T_\alpha$ and $t(\alpha) \in T_\beta$ such that $s(\alpha) \in A_{t(\alpha)}$. If we define $R := \bigcup_{t \in T_\beta} A_t$, then we have

$$\{s(\alpha) \mid \alpha \in Y\} \subseteq R,$$

and consequently $Y \subseteq \widehat{R} = \bigcup_{t \in T_\beta} \widehat{A_t}$. However, $\widehat{R}$ is nonstationary (being a countable union of nonstationary sets) and contains the stationary set Y, which is a contradiction. $\quad\quad _{\text{Claim}}\square \quad\quad\quad \square$

Theorem 4.13. *If T is an almost-Suslin tree, then $\Diamond_T$ holds if and only if $\Diamond$ holds.*

Proof. $\Diamond_T$ implies $\Diamond$ by Theorem 3.5, and $\Diamond$ implies $\Diamond_T$ by Lemmas 4.8 and 4.11. $\quad\quad\quad \square$

In particular:

Corollary 4.14. *If T is a Suslin tree, then $\Diamond_T$ holds if and only if $\Diamond$ holds.*

The trees that arise in the study of walks on ordinals (see [31]) are important examples of ω_1-trees. It turns out that the trees $T(\rho_0)$ and $T(\rho_2)$ are always special (see [31] and [25]). However, the situation for ρ_1 is different: $T(\rho_1)$ can be nonspecial and can even be almost-Suslin. There are two different ways to see this.

Todorcevic showed that given a ladder system $\overrightarrow{C}$ (see Definition 6.5) and $r \in ([\omega]^{<\omega})^\omega$, one can associate another sequence $\overrightarrow{C}^r$ which, for suitable r, is again a ladder system. He proved that if r is a Cohen real then the ρ_1-tree associated to $\overrightarrow{C}^r$ is almost-Suslin (see [34, Theorem 15]).

A second approach uses the combinatorial principle $\bigstar_1$, which asserts the existence of a ladder system $\overrightarrow{C} = (C_\alpha \mid \alpha \in \mathrm{LIM}(\omega_1))$ with the property that for every stationary $S \subseteq \omega_1$ there exist ordinals $\alpha, \beta \in S \cap \mathrm{LIM}(\omega_1)$ such that $C_\beta \cap \alpha \sqsubseteq C_\alpha$ and $\alpha \in C_\beta$. Hrušák and Martínez Ranero proved that if $\overrightarrow{C}$ is a $\bigstar_1$-sequence then the ρ_1-tree associated to $\overrightarrow{C}$ is almost-Suslin (see [14, Theorem 4]).

Therefore it is consistent that $T(\rho_1)$ is almost-Suslin, in particular, by Theorem 4.14, we obtain:

Corollary 4.15. *It is consistent that $\Diamond_{T(\rho_1)}$ holds.*

Since coherent trees tend to be $\leq$-comparable, the preceding result naturally leads to the question of what can be said about $\Diamond_T$ for this class of trees.

5. $\Diamond^*$ is stronger than "$\Diamond_T$ holds for every nonspecial Aronszajn tree"

We already know that for every nonspecial, well-pruned ω_1-tree, $\Diamond^*$ implies $\Diamond_T$ (see Corollary 4.3). An interesting question, then, is whether it is consistent with $\neg\Diamond^*$ that $\Diamond_T$ holds for every well-pruned Aronszajn tree T. The first part of this section is devoted to providing a positive answer to this question.

Theorem 5.1. *It is consistent with $\neg\Diamond^*$ that there exist nonspecial, well-pruned Aronszajn trees and that $\Diamond_T$ holds for all such trees.*

The proof of Theorem 5.1 proceeds via a countable-support iteration of length ω_2 over a model of CH. We alternate forcing with $2^{<\omega_1}$—which destroys $\Diamond^*$-sequences—and the Jensen–Kurepa forcing, which introduces $\Diamond^+$-sequences. Although these two forcings have opposing effects, both are σ-closed; therefore, by Baumgartner's theorem (see Theorem 5.6), under CH the iteration satisfies the ω_2-chain condition. In particular, every nonspecial ω_1-tree T in $V[G]$ appears at some intermediate stage of the iteration. After that stage we add a $\Diamond^+$-sequence, which implies $\Diamond_T$ at that step. Moreover, the

remainder of the iteration is σ-closed, so it neither specializes trees nor destroys $\Diamond_T$-sequences (see Lemmas 4.6 and 5.2); hence $\Diamond_T$ holds in the final model $V[G]$. On the other hand, also by the ω_2-c.c., every $\Diamond^*$-candidate $\overrightarrow{\mathcal{D}}$ in $V[G]$ already appears at some intermediate stage; after that stage we force with $2^{<\omega_1}$, ensuring that $\overrightarrow{\mathcal{D}}$ is no longer a $\Diamond^*$-sequence. Since the remainder of the iteration is proper, $\overrightarrow{\mathcal{D}}$ remains non-$\Diamond^*$ in $V[G]$.

Thus we proceed by iterating σ-closed forcing notions over a model of CH. By Lemma 4.6 such forcings do not specialize ω_1-trees; we now show that they also preserve $\Diamond_T$-sequences.

Lemma 5.2. *Let T be a nonspecial ω_1-tree and $(D_t)_{t \in T}$ a $\Diamond_T$-sequence. If $\mathbb{P}$ is a σ-closed forcing, then $\mathbb{P} \Vdash$ "$(D_t)_{t \in T}$ is a $\Diamond_T$-sequence".*

Proof. Let $p \in \mathbb{P}$ and $\dot{X}, \dot{B}$ be such that $p \Vdash$ "$\dot{X} \subseteq \check{T} \wedge \dot{B} \in NS^{\check{T}}$". We aim to show that there exists $q \leq p$ such that $q \Vdash$ "$\exists t \in T \setminus \dot{B}(\dot{X} \cap \check{t}{\downarrow} = \check{D}_t)$".

Now, $\dot{B} = \nabla_{s \in T} \dot{A}_s$, where each $\dot{A}_s$ is a $\mathbb{P}$-name for a special subset of T, there exists a $\mathbb{P}$-name $\dot{f}_s$ for a specializing function of $\dot{A}_s$, i.e., $\dot{f}_s : \dot{A}_s \to \omega$.

For every $s, t \in T$, define

$$D(s,t) = \{q \in \mathbb{P} \mid q \text{ decides if } t \in \dot{A}_s, \text{ and if } q \Vdash \text{``}t \in \dot{A}_s\text{''}, \text{ then } q \text{ decides } \dot{f}_s(t)\}.$$

Note that $D(s,t)$ is dense in $\mathbb{P}$ for every $(s,t) \in T^2$.

For every $s \in T$, let $E(s)$ denote the set $\{q \in \mathbb{P} \mid q \text{ decides } s \in \dot{X}\}$, which is also dense in $\mathbb{P}$.

Since $\mathbb{P}$ is σ-closed, there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D(s,t) \neq \emptyset \neq G \cap E(s)$ for every $(s,t) \in T^2$.

For each $\mathbb{P}$-name $\dot{x}$, we can evaluate it with respect to the filter G, namely $\dot{x}[G]$, although G need not be generic. Thus, for every $s \in T$ consider $\dot{A}_s[G]$ and $\dot{f}_s[G]$, both of which belong to V since $G \in V$. Now set

$$\hat{B} := \nabla_{s \in T}\big(\dot{A}_s[G]\big),$$

and note that $\hat{B} \in NS^T$ because each $\dot{A}_s[G]$ is special, as witnessed by $\dot{f}_s[G]$.

Next, construct a decreasing sequence $(q_\alpha)_{\alpha \in \omega_1}$ such that $q_0 \leq p$ and, for every $\alpha \in \omega_1$, we have

$$q_\alpha \Vdash \text{``}\Big(\dot{B} \cap T_{<\alpha} = \hat{B} \cap T_{<\alpha}\Big) \wedge \Big(\dot{X} \cap T_{<\alpha} = \dot{X}[G] \cap T_{<\alpha}\Big)\text{''}.$$

Since $\hat{B} \in NS^T$, there exists $t \in T$ such that $t \notin \hat{B}$ and $\dot{X}[G] \cap t{\downarrow} = D_t$. Let $\alpha = \text{ht}(t) + 1$, and note that q_α satisfies:

- $q_\alpha \Vdash$ "$t \notin \hat{B}$", and thus $q_\alpha \Vdash$ "$t \notin \dot{B}$" because $q_\alpha \Vdash$ "$\dot{B} \cap T_{<\alpha} = \hat{B} \cap T_{<\alpha}$".
- $q_\alpha \Vdash$ "$\dot{X} \cap t{\downarrow} = D_t$", since $q_\alpha \Vdash$ "$\dot{X} \cap T_{<\alpha} = \dot{X}[G] \cap T_{<\alpha}$" and $q_\alpha \Vdash$ "$\dot{X}[G] \cap t{\downarrow} = D_t$" (this last is because $\dot{X}[G] \cap t{\downarrow} = D_t$ holds in V).

Hence, $q_\alpha \Vdash$ "$t \notin \dot{B} \wedge (\dot{X} \cap t{\downarrow} = D_t)$", which completes the proof. $\qquad\square$

If $\overrightarrow{\mathcal{D}} = (\mathcal{D}_\alpha)_{\alpha \in \omega_1}$ is such that $\mathcal{D}_\alpha \subseteq [\alpha]^{\leq \omega}$ for every $\alpha \in \omega_1$, then let us say that $\overrightarrow{\mathcal{D}}$ is a $\Diamond^*$-*candidate*.

The proof of the following result, which appears as Exercise (J5) in [21, p. 300], follows essentially the same argument given in [28, p. 387], and it also appears in Devlin's proof that $\diamondsuit$ does not imply $\diamondsuit^*$ [7, Theorem 3.2].

Proposition 5.3. *Let $\overrightarrow{\mathcal{D}}$ be a $\diamondsuit^*$-candidate and $\mathbb{P} = 2^{<\omega_1}$. Then $\mathbb{P} \Vdash$ "$\overrightarrow{\mathcal{D}}$ is not a $\diamondsuit^*$-sequence."*

The following two forcing notions will play an important role in the proof of Theorem 5.1.

The *Jech-Suslin forcing* $\mathbb{JS}$ is defined as: $p \in \mathbb{JS}$ if and only if $p = \emptyset$ or p is a subtree of 2^{ω_1} such that:

   (i) $\mathrm{ht}(p) = \alpha + 1$ for some $\alpha \in \mathrm{LIM}(\omega_1)$,
   (ii) For all $s \in p$ such that $\mathrm{dom}(s) < \alpha$ we have that $s^\frown 0, s^\frown 1 \in p$,
   (iii) For all $\xi \leq \alpha$ we have $|p_\xi| \leq \omega$ and
   (iv) For all $s \in p$ there exists $t \in p_\alpha$ such that $s \subseteq t$.

And $p \leq q$ if $q = \{s \in p \mid \mathrm{ht}(s) < \mathrm{ht}(q)\}$, that is, if $q = p \upharpoonright \mathrm{ht}(q)$. The reason we call $\mathbb{JS}$ the *Jensen-Suslin forcing* is that Jech introduced it to force the existence of a Suslin-tree (see [21, Exercices H11, p. 248])

Note that if p, q are two compatible elements in $\mathbb{JS}$ then $p \leq q$ or $q \leq p$. In fact, if $\mathrm{ht}(q) \leq \mathrm{ht}(p)$ and $r \leq p, q$ then:

$$q = r \upharpoonright \mathrm{ht}(q) = (r \upharpoonright \mathrm{ht}(p)) \upharpoonright \mathrm{ht}(q) = p \upharpoonright \mathrm{ht}(q)$$

and thus $p \leq q$. It is also easy to see that $|\mathbb{JS}| = 2^\omega$, so if CH holds, then $|\mathbb{JS}| = \omega_1$.

The *Jensen-Kurepa forcing* $\mathbb{JK}$ is defined as follows: $(p, I) \in \mathbb{JK}$ if the following hold:

   (1) $p \in \mathbb{JS}$ and $I \subseteq 2^{\omega_1}$ such that $|I| \leq \omega$,
   (2) $f \upharpoonright (\mathrm{ht}(p) - 1) \in p$ for all $f \in I$.

and $(p, I) \leq (q, J)$ if $p \leq q$ in $\mathbb{JS}$ and $I \supseteq J$. Analogously to the case of $\mathbb{JS}$, the reason we call $\mathbb{JK}$ *Jensen-Kurepa* is that Jensen introduced it to force the existence of a Kurepa-tree under CH (see [21, Exercices H18 and H19, p. 249])

Recall that a subset R of a forcing $\mathbb{P}$ is *centered* if every finite subset of R has a lower bound in $\mathbb{P}$, and it is *linked* if every pair of elements of R has a lower bound in $\mathbb{P}$ (in both cases the bound need not lie in R). We say that $\mathbb{P}$ is ω_1-*centered* (respectively, ω_1-*linked*) if it can be expressed as the union of ω_1 many centered (respectively, linked) subsets.

Note that if we have two conditions in $\mathbb{JK}$ with the same stem, namely (p, I) and (p, J), then $(p, I \cup J)$ is a common lower bound. This fact, together with the observation that $|\mathbb{JS}| = 2^\omega$, shows that if CH holds, then $\mathbb{JK}$ is ω_1-centered; in particular, it is ω_1-linked. Another important consequence of CH about $\mathbb{JK}$ is the following:

Proposition 5.4. [21, Exercice H20, p. 249] *If CH holds, then $\mathbb{JK} \Vdash$ "$\diamondsuit^+$".*

On the other hand, an important property of $\mathbb{JK}$ is that it is σ-closed. In fact, let $(p_n, I_n)_{n \in \omega}$ be a decreasing sequence in $\mathbb{JK}$, and for every $n \in \omega$ let $\alpha_n \in \mathrm{LIM}(\omega_1)$ be the unique ordinal such that $\alpha_n + 1 = \mathrm{ht}(p_n)$; note that $\alpha_n \leq \alpha_{n+1}$ for all $n \in \omega$.

If the sequence $(\alpha_n)_{n \in \omega}$ is eventually constant, say with constant value α_m, then $p_n = p_m$ for every $n \geq m$, and consequently $(p_m, \bigcup_{n \in \omega} I_n)$ is a lower bound for $(p_n, I_n)_{n \in \omega}$. Now, suppose that $(\alpha_n)_{n \in \omega}$ is not eventually constant and let $\alpha = \lim_{n \in \omega} \alpha_n$. Consider the set

$$S = \{s \in 2^{<\omega_1} \mid (\exists m \in \omega)(s \in p_m) \wedge (\forall f \in \bigcup_{n \in \omega} I_n)(s \nsubseteq f)\}.$$

It is clear that $|S| \leq \omega$, and by condition (iv) of the definition of Jech-Suslin forcing, for every $s \in S$ there exists a (cofinal) branch $g \in [\bigcup_{n \in \omega} p_n]$ such that $s \subseteq g$. For each $s \in S$, fix one such branch g_s. Now, define

$$r = \left(\{f \restriction \beta \mid (\exists n \in \omega)(f \in I_n) \wedge (\beta \leq \alpha)\}\right) \cup \left(\bigcup_{n \in \omega} p_n\right) \cup \{g_s \mid s \in S\}.$$

We claim that $(r, \bigcup_{n \in \omega} I_n)$ is in $\mathbb{JK}$.

It clearly suffices to show that $r \in \mathbb{JS}$. Conditions (i) and (iv) hold by construction, while condition (iii) also holds because $|r_\alpha| \leq \omega$ and $|r_\xi| \leq |r_\alpha|$ by condition (iv). To verify condition (ii), note that for every $s \in r$ with $\mathrm{ht}(s) < \alpha$, there exists some $n \in \omega$ such that $s \in p_n$. Indeed, the only nontrivial case occurs when $s = f \restriction \beta$ for some $\beta < \alpha$ and some $f \in I_n$ for some $n \in \omega$. Since $(\alpha_n)_{n \in \omega}$ is (eventually) increasing, there exists $m > n$ such that $\alpha_m > \beta$, and because $I_n \subseteq I_m$, we have $f \restriction \alpha_m \in p_m$. Consequently, $s = f \restriction \beta \in p_m$, and hence both $s^\frown 0$ and $s^\frown 1$ belong to $p_m \subseteq r$, which completes the verification. The last argument also shows that $r \restriction \alpha_m + 1 = p_m$, so we are done.

An important definition for proving Theorem 5.1 is the following:

Definition 5.5. A forcing $\mathbb{P}$ is called *well-met* if every two compatible conditions have an infimum in $\mathbb{P}$.

It is clear that $2^{<\omega_1}$ is well-met since if $p, q \in 2^{<\omega_1}$ then either $p \leq q$ or $q \leq p$. It is also true that $\mathbb{JK}$ is well-met. To see this, suppose that (p, I) and (q, J) are compatible. In particular, as mentioned before, we must have either $p \leq q$ or $q \leq p$. Without loss of generality, assume $p \leq q$. We claim that $(p, I \cup J)$ is a condition in $\mathbb{JK}$.

Indeed, since (p, I) and (q, J) are compatible in $\mathbb{JK}$, there exists some $(r, K) \leq (p, I), (q, J)$. In particular, $J \subseteq K$, and therefore for every $f \in J$ we have

$$f \restriction (\mathrm{ht}(r) - 1) \in r.$$

But since $p = r \restriction \mathrm{ht}(p)$, it follows that

$$f \restriction (\mathrm{ht}(p) - 1) \in p.$$

Thus, $(p, I \cup J)$ is a condition. Moreover, $(p, I \cup J)$ is the infimum of (p, I) and (q, J), since if (r', K) is a lower bound for both, then necessarily $r' \leq p$ and $K \supseteq I \cup J$.

The following theorem by Baumgartner [3] (which also appears as [22, Lemma V.5.14, p. 360]) is the final piece we need to prove Theorem 5.1:

Theorem 5.6. Let $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \mid \beta \in \alpha \rangle$ be a countable support iteration of forcing notions that are σ-closed, well-met and ω_1-linked. If CH holds, then $\mathbb{P}_\alpha$ satisfies the ω_2-chain condition.

Theorem 5.6 is central to the study of the forcing axiom BACH, which yields significant topological and set-theoretic consequences (see [29]).

Proof of Theorem 5.1. Start with a model of GCH and consider $\mathbb{P} = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \mid \beta \in \omega_2 \rangle$ a countable support iteration of forcing notions such that at even stages we use $2^{<\omega_1}$ and at odd stages we use $\mathbb{JK}$. That is, if $\beta \in \omega_2$ is even, then

$$\mathbb{P}_\beta \Vdash \text{``}\dot{\mathbb{Q}}_\beta = 2^{<\omega_1}\text{''}$$

and if β is odd, then

$$\mathbb{P}_\beta \Vdash \text{``}\dot{\mathbb{Q}}_\beta = \mathbb{JK}\text{''}.$$

Now, let G be $\mathbb{P}$-generic filter. By Theorem 5.6, assuming CH, the iteration $\mathbb{P}$ has the ω_2-cc. In particular, any $\mathbb{P}$-name $\dot{\vec{\mathcal{D}}}$ for a $\Diamond^*$-candidate is decided at some stage γ. Since at that stage or the next we force with $2^{<\omega_1}$, Proposition 5.3 implies that $\mathbb{P}_{\gamma+1} \Vdash \text{``}\dot{\vec{\mathcal{D}}}$ is not a $\Diamond^*$-sequence.''. Hence in $V[G \upharpoonright \gamma + 1]$ there is $X \subseteq \omega_1$ such that $\{\alpha < \omega_1 \mid X \cap \alpha \in \dot{\mathcal{D}}_\alpha[G \upharpoonright \gamma + 1]\}$ fails to contain a club. Equivalently, $\{\alpha < \omega_1 \mid X \cap \alpha \notin \dot{\mathcal{D}}_\alpha[G \upharpoonright \gamma + 1]\}$ is stationary. Since each iterand in the remainder of the iteration is proper, the tail forcing is proper and thus this stationary set remains stationary in $V[G]$. Therefore $\mathbb{P} \Vdash \text{``}\dot{\vec{\mathcal{D}}}$ is not a $\Diamond^*$-sequence''. Finally, because the argument applies to every $\mathbb{P}$-name $\dot{\vec{\mathcal{D}}}$, we conclude

$$\mathbb{P} \Vdash \text{``}\neg \Diamond^*\text{.''}$$

On the other hand, since $\mathbb{P}$ has the ω_2-cc, any $\mathbb{P}$-name $\dot{T}$ for an ω_1-tree in $V[G]$ is decided at some stage η. By Lemma 4.6, if $\dot{T}[G \upharpoonright \eta]$ is nonspecial then it remains nonspecial in $V[G]$; in particular, every nonspecial Aronszajn tree from V stays nonspecial in $V[G]$. Moreover, because we force cofinally often with $\mathbb{JK}$ (which adds $\Diamond^+$ and hence $\Diamond^*$), Corollary 4.3 shows that $\Diamond_T$ holds in $V[G \upharpoonright (\eta + 1)]$. Finally, each remaining iterand is σ-closed, so the tail forcing is σ-closed; therefore, by Lemma 5.2, $\Diamond_T$ continues to hold in $V[G]$. $\qquad\square_{5.1}$

6. $\Diamond$ MAY BE STRICTLY WEAKER THAN SOME $\Diamond_T$

We already know that if T is a nonspecial ω_1-tree then $\Diamond_T$ is strictly weaker than $\Diamond^*$ and is at least as strong as $\Diamond$; in fact, if T is Suslin then $\Diamond_T$ is equivalent to $\Diamond$. Thus, the remaining question is whether there exists (or may exist) a nonspecial ω_1-tree such that $\Diamond$ is strictly weaker than $\Diamond_T$. This entire section is dedicated to give a positive answer to this.

Theorem 6.1. *It is consistent with $\Diamond$ that there exists a nonspecial Aronszajn tree T such that $\Diamond_T$ does not hold.*

The plan to prove Theorem 6.1 is as follows:

(1) Start with $R \subseteq \omega_1$ being bistationary (i.e., both R and $\omega_1 \setminus R$ are stationary) and a model of $\Diamond^*(\omega_1 \setminus R)$ (for example $V = L$).

(2) Thanks to Theorem 6.4, we have a nonspecial Aronszajn tree T such that $T \upharpoonright R \in NS^T$ and $T \upharpoonright (\omega_1 \setminus R) \notin NS^T$.

(3) Iterate the forcing from Definition 6.7 to kill $\clubsuit(\omega_1 \setminus R)$[11]. This is carried out in Proposition 6.8 and Theorem 6.23[12]

(4) Prove that this iteration of forcing preserves the following:
   (a) $\Diamond(R)$ (Theorem 6.23 applied to $S := \omega_1 \setminus R$)),
   (b) $T \restriction (\omega_1 \setminus R) \notin NS^T$ (By Theorem 6.29, Theorem 6.33 and Proposition 6.27).

(5) Apply Theorem 3.3 to argue that $\Diamond_T$ does not hold in $V[G]$, since in this case $\Diamond(\omega_1 \setminus R)$ would hold in $V[G]$, but not even $\clubsuit(\omega_1 \setminus R)$ holds in $V[G]$.

(6) Then such a model is a model of $\Diamond(R) + \neg\Diamond_T$, in particular a model of $\Diamond + \neg\Diamond_T$.

We will begin by recalling an important theorem of Shelah. The theorem essentially states that, under certain hypothesis, given a bistationary set $S \subseteq \omega_1$, there exists a nonspecial Aronszajn tree T such that $T \restriction S \in NS^T$. Although this theorem is originally stated in different terms, it is crucial for the proof of Theorem 6.1, since, as previously suggested, such a tree will serve—after a forcing extension—as the witness for the theorem. As Shelah's theorem is formulated using different terminology, we will first introduce a preliminary definition and lemma in order to properly state it.

Definition 6.2. [26, Definition 3.3, p. 444] Let $S \subseteq \omega_1$ be stationary, and let T be an ω_1-tree. We say that T is *S-st-special* if there exists a function f such that:

(1) $\mathrm{dom}(f) = T \restriction S \setminus \{0\}$,
(2) if $t \in T_\alpha$ for some $\alpha \in S \setminus \{0\}$, then $f(t) \in \alpha \times \omega$, and
(3) if $s, t \in \mathrm{dom}(f)$ are such that $s < t$, then $f(s) \neq f(t)$.

It turns out that if T is S-st-special for some $S \subseteq \omega_1$ stationary, then T is Aronszajn but not Suslin [26, Claim 3.4(1), p. 445]. On the other hand, remarkably, Shelah's previous notion of S-st-special can be expressed in Brodsky's terminology:

Proposition 6.3. *Let T be an ω_1-tree and let $S \subseteq \omega_1$ be stationary. Then T is S-st-special if and only if $T \restriction S \in NS^T$.*

Proof. Suppose that T is S-st-special, and let $f = (f_0, f_1)$ be a witness of this[13]. Now, let $g \colon T \restriction (S \setminus \{0\}) \to T$ be given by $g(s) = s \restriction f_0(s)$. Clearly, g is regressive. Let us now show that $g^{-1}[\{t\}]$ is special for every $t \in T$. Note that:

$$g^{-1}[\{t\}] = \{s \in \mathrm{dom}(f) \mid (s > t) \wedge (f_0(s) = \mathrm{ht}(t))\} = \bigcup_{n \in \omega} \{s \in \mathrm{dom}(f) \mid (s > t) \wedge (f(s) = (\mathrm{ht}(t), n))\}.$$

If $s_0, s_1 > t$ are such that $f(s_0) = (\mathrm{ht}(t), n) = f(s_1)$, then necessarily s_0 and s_1 are incomparable, which proves that $g^{-1}[\{t\}]$ is special. This proves that $T \restriction (S \setminus \{0\}) \in NS^T$ and consequently $T \restriction S \in NS^T$.

Conversely, suppose that $T \restriction S \in NS^T$. Then, there exists a regressive function $g : T \restriction S \to T$ such that $g^{-1}[\{t\}]$ is special for every $t \in T$. That is, we can write

$$g^{-1}[\{t\}] = \bigcup_{n \in \omega} A_n^t,$$

---

[11]See Definition 6.6; for now it suffices to know that $\Diamond(\omega_1 \setminus R)$ implies $\clubsuit(\omega_1 \setminus R)$.
[12]These results are applied with $S = \omega_1 \setminus R$.
[13]Here, we are using the notation $f(t) = (f_0(t), f_1(t))$ for every $t \in T \restriction S \setminus \{0\}$.

where each A_n^t is an antichain. Now, let us define a function f as follows:

(1) $\mathrm{dom}(f) = T \upharpoonright S \setminus \{0\}$,

(2) if $s \in g^{-1}[\{t\}]$, then $f(s) = (\mathrm{ht}(t), n)$, where $n \in \omega$ is such that $s \in A_n^t$.

It should be clear that f witnesses that T is S-st-special. $\qquad\square$

Theorem 6.4. [26, Lemma 3.9, p. 448] *Let $S \subseteq \omega_1$ be stationary and assume that $\Diamond^*(\omega_1 \setminus S)$ holds. Then there is a well-pruned Aronszajn tree T such that $T \upharpoonright S \in NS^T$ and for all $R \subseteq \omega_1$ we have that $T \upharpoonright R \in NS^T$ if and only if $R \setminus S$ is nonstationary. Moreover, there is no antichain $A \subseteq T$ such that $\widehat{A} \setminus S$ is stationary.*

The fundamental forcing notion that we will work with in the remainder of this article for the proof of Theorem 6.1 is the forcing designed to destroy $\clubsuit(S)$ sequences.

Definition 6.5. Let $S \subseteq \omega_1$ be a stationary set. A sequence $\overrightarrow{C} = (C_\alpha \mid \alpha \in S \cap \mathrm{LIM}(\omega_1))$ is called a *ladder system in S* if $C_\alpha \subseteq \alpha$ is cofinal of order type ω for every $\alpha \in S \cap \mathrm{LIM}(\omega_1)$.

Definition 6.6. Let $S \subseteq \omega_1$ be a stationary set and $\overrightarrow{C} = (C_\alpha \mid \alpha \in S \cap \mathrm{LIM}(\omega_1))$ a ladder system on S. Then $\overrightarrow{C}$ is called a *$\clubsuit(S)$-sequence* if for every $X \in [\omega_1]^{\omega_1}$, the set $\{\alpha \in S \cap \mathrm{LIM}(\omega_1) \mid C_\alpha \subseteq X \cap \alpha\}$ is stationary.

Definition 6.7. Let $S \subseteq \omega_1$ be stationary and $\overrightarrow{C} = (C_\alpha \mid \alpha \in S \cap \mathrm{LIM}(\omega_1))$ be a ladder system on S. The forcing $\mathbb{P}(S, \overrightarrow{C})$ is defined as follows: $p \in \mathbb{P}(S, \overrightarrow{C})$ if and only if there is some $\alpha \in \omega_1$ such that $p : \alpha \to 2$, and for every $\beta \in (\alpha + 1) \cap S \cap \mathrm{LIM}(\omega_1)$, we have $C_\beta \not\subseteq p^{-1}(\{1\})$. The order is given by $p \leq q$ if $p \supseteq q$.

The forcing $\mathbb{P}(S, \vec{C})$ has two desirable features: it is proper and ω-distributive, hence totally proper[14]. Indeed, by Theorem 7.13 of [9], the poset is even completely proper and $<\omega_1$-proper via the fusion schemes ($\mathrm{CP}_{\aleph_1}$) and (A). It also has two additional properties crucial for our goals: it is $\sigma(\omega_1 \setminus S)$-closed (Definition 6.12), and it is strategically closed in models (Definition 6.26). In fact, we will give an alternative proof—using strategic closure in models—of its total properness (see Proposition 6.27 and Theorem 6.28). However, the most important feature—and the very reason we employ this forcing—is that it is designed to destroy the sequence $\overrightarrow{C}$ as an instance of $\clubsuit(S)$:

Proposition 6.8. *Let $S \subseteq \omega_1$ be stationary and $\overrightarrow{C} = (C_\alpha \mid \alpha \in S \cap \mathrm{LIM}(\omega_1))$ be a ladder system on S. Then $\mathbb{P}(S, \overrightarrow{C}) \Vdash$ "$\overrightarrow{C}$ is not a $\clubsuit(S)$-sequence".*

Proof. For every $\alpha \in \omega_1$, let $D_\alpha = \{p \in \mathbb{P}(S, \overrightarrow{C}) \mid \exists \beta \exists \gamma (\alpha \leq \gamma < \beta (p : \beta \to 2 \wedge p(\gamma) = 1))\}$.

Claim 6.9. D_α is dense for every $\alpha \in \omega_1$.

Proof of the Claim. Let $p \in \mathbb{P}(S, \overrightarrow{C})$ and let $\xi = \mathrm{dom}(p)$. Now, let $\beta \in S \cap \mathrm{LIM}(\omega_1)$ be such that $\beta > \alpha, \xi$ and enumerate $(\xi, \beta]$ as $\{\alpha_n \mid n \in \omega\}$ such that $\alpha_0 = \beta$. Now construct recursively $\{a_n \mid n \in \omega\}$ and $\{b_n \mid n \in \omega\}$ as follows:

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[14]Recall that a forcing is *totally proper* if it is proper and does not add new reals.

(1) $a_0, b_0 \in C_{\alpha_0} \setminus \xi$ such that $\alpha < a_0 < b_0$.

(2) $a_{n+1}, b_{n+1} \in C_{\alpha_{n+1}} \setminus (\xi \cup \{a_i, b_i \mid i \leq n\}$ and $a_{n+1} < b_{n+1}$.

Now let $q : \beta \to 2$ be given by:

$$q(\delta) = \begin{cases} p(\delta) & \text{if } \delta \in \xi, \\ 0 & \text{if } \exists n \in \omega(\delta = a_n), \\ 1 & \text{if } \exists n \in \omega(\delta = b_n), \\ 0 & \text{in other case.} \end{cases}$$

Clearly $q \in D_\alpha$ and $q \leq p$. $Claim\square$

Let us go to $V[G]$. Let $f_G := \cup G : \omega_1 \to 2$. Then, since every D_α is dense, $X_G := f_G^{-1}(\{1\})$ is cofinal in ω_1 (and $\omega_1^V = \omega_1^{V[G]}$, which follows from either the properness of $\mathbb{P}(S, \vec{C})$ or its ω-distributivity).

Claim 6.10. $C_\alpha \not\subseteq X_G$ for every $\alpha \in \omega_1$.

Proof of the Claim. If $C_\alpha \subseteq X_G$ for some $\alpha \in \omega_1$, then $C_\alpha \subseteq X_G \cap \alpha$ as $C_\alpha \subseteq \alpha$. Now, let $p \in G$ be such that $\mathrm{dom}(p) > \alpha$. Then p is such that $X_G \cap \alpha \subseteq p^{-1}(\{1\})$. Thus, we have that:

$$C_\alpha \subseteq X_G \cap \alpha \subseteq p^{-1}(\{1\}),$$

which is impossible as $p \in \mathbb{P}(S, \vec{C})$. $Claim\square$ $\square$

The next step in the proof of Theorem 6.1 is to prove that $\Diamond(\omega_1 \setminus S)$ is preserved under iterations of forcings of the form $\mathbb{P}(S, \vec{C})$, where $\vec{C}$ ranges over all possible ladder systems on S. To this end, we begin by proving that $\Diamond$ implies the existence of a seemingly stronger sequence. The following is based on a result by Hrušák and the first author [11, Lemma 4.9] (see also [15]).

Lemma 6.11. *Let $S \subseteq \omega_1$ be stationary, $\mathbb{P}$ be a forcing notion, κ be a large enough regular cardinal and assume $V \models \Diamond(S)$. Then there is a sequence $\langle (M_\alpha, p_\alpha, \dot{X}_\alpha) \mid \alpha \in S \cap \mathrm{LIM}(\omega_1) \rangle$ such that for every $\alpha \in S \cap \mathrm{LIM}(\omega_1)$, the following holds:*

(1) M_α is a countable elementary submodel of $\mathsf{H}(\kappa)$ such that $\mathbb{P}, p_\alpha, \dot{X}_\alpha \in M_\alpha$.

(2) $p_\alpha \in \mathbb{P}$ and $p_\alpha \Vdash "\dot{X}_\alpha \subseteq \omega_1"$.

The sequence $\langle (M_\alpha, p_\alpha, \dot{X}_\alpha) \mid \alpha \in S \cap \mathrm{LIM}(\omega_1) \rangle$ has the property that for every $p \in \mathbb{P}$ and $\dot{X}$ such that $p \Vdash "\dot{X} \subseteq \omega_1"$, there is a countable $N \preceq \mathsf{H}(\kappa)$ and $\alpha \in S \cap \mathrm{LIM}(\omega_1)$ such that the following conditions hold:

(I) $\mathbb{P}, p, \dot{X} \in N$.

(II) $M_\alpha \cap \omega_1 = \alpha$.

(III) The structures $(N, \in, \mathbb{P}, \leq_\mathbb{P}, \Vdash_\mathbb{P}, p, \dot{X})$ and $(M_\alpha, \in, \mathbb{P}, \leq_\mathbb{P}, \Vdash_\mathbb{P}, p_\alpha, \dot{X}_\alpha)$ are isomorphic.

Proof. Using $\Diamond(S)$ we can find a sequence

$$\langle \mathfrak{A}_\alpha = (\alpha, \triangle_\alpha, P_\alpha, \precsim_\alpha, \rightsquigarrow_\alpha, r_\alpha, Y_\alpha) \mid \alpha \in S \cap \mathrm{LIM}(\omega_1) \rangle$$

such that for every structure $\mathfrak{A} = (\omega_1, \triangle, P, \precsim, \rightsquigarrow, r, Y)$ –with $\triangle, \precsim, \rightsquigarrow \subseteq \omega_1{}^2$, $P, Y \subseteq \omega_1$ and $r \in \omega_1$– there are stationary many $\alpha \in S$ such that $\mathfrak{A}_\alpha$ is an elementary substructure of $\mathfrak{A}$. Given α a limit ordinal in S, in case there are a countable $M \preceq \mathsf{H}(\kappa)$, $p \in \mathbb{P}$, $\dot{X}$ such that

(i) $\mathbb{P}, \leq_\mathbb{P}, p, \dot{X} \in M$,

(ii) $M \cap \omega_1 = \alpha$,

(iii) $p \Vdash ``\dot{X} \subseteq \omega_1"$ and

(iv) $(M, \in, \mathbb{P}, \leq_{\mathbb{P}}, \Vdash_{\mathbb{P}}, p, \dot{X})$ is isomorphic to $\mathfrak{A}_\alpha$,

then we choose one of them and define $M_\alpha = M$, $p_\alpha = p$ and $\dot{X}_\alpha = \dot{X}$. If there is no M that satisfies these properties, we just take any $(M_\alpha, p_\alpha, \dot{X}_\alpha)$ that satisfies the properties (1) and (2) of the Lemma.

We will now prove that $\langle (M_\alpha, p_\alpha, \dot{D}_\alpha) \mid \alpha \in S \cap \mathrm{LIM}(\omega_1) \rangle$ has the desired properties.

Let $p \in \mathbb{P}$ and $\dot{X}$ be a $\mathbb{P}$-name such that $p \Vdash ``\dot{X} \subseteq \omega_1"$. Recursively, we build $\{N_\alpha \mid \alpha < \omega_1\}$ a continuous $\in$-chain of countable elementary submodels of $\mathsf{H}(\kappa)$ such that $p, \dot{X}, \mathbb{P} \in N_0$. Let $N = \bigcup_{\alpha < \omega_1} N_\alpha$, since N has size ω_1, then we can define a structure

$$\mathfrak{A} = (\omega_1, \triangle, P, \leq, \rightsquigarrow, r, Y)$$

that is isomorphic to $(N, \in, \mathbb{P}, \leq_{\mathbb{P}}, \Vdash, p, \dot{X})$. Let $F : \omega_1 \to N$ be an isomorphism.

It is easy to see that

$$\{\alpha \in \mathrm{LIM}(\omega_1) \mid N_\alpha \cap \omega_1 = \alpha \wedge F[\alpha] = N_\alpha\}$$

is a club. In this way, we can find $\alpha \in S \cap \mathrm{LIM}(\omega_1)$ such that $F[\alpha] = N_\alpha$, $N_\alpha \cap \omega_1 = \alpha$ and $\mathfrak{A}_\alpha \preceq \mathfrak{A}$. Note that N_α, p, and $\dot{X}$ satisfy conditions (i), (ii), (iii), and (iv) at stage α. In particular, condition (iv) holds, showing that $\mathfrak{A}_\alpha$ is isomorphic to both

$$(N, \in, \mathbb{P}, \leq_{\mathbb{P}}, \Vdash_{\mathbb{P}}, p, \dot{X}) \quad \text{and} \quad (M_\alpha, \in, \mathbb{P}, \leq_{\mathbb{P}}, \Vdash_{\mathbb{P}}, p_\alpha, \dot{X}_\alpha).$$

Hence those two structures are isomorphic to each other. $\square$

The following definition captures what we consider to be one of the essential properties of forcings of the form $\mathbb{P}(S, \overrightarrow{C})$, and this property ensures that the forcings possessing it preserve certain diamond principles (see Theorem 6.14 and Proposition 6.21). We must clarify that this appears to be a specific case of the theory that Shelah develops in [26, Chapter V, Section 1]; however, in our opinion, that presentation is very technical and hard to follow, which is why we decided to include this material, a presentation that we believe is quite accessible.

Definition 6.12. Let $(\mathbb{P}, \leq_{\mathbb{P}})$ be a forcing, κ a cardinal, $M \preceq \mathsf{H}(\kappa)$ with $\mathbb{P} \in M$ and $(p_n)_{n \in \omega}$ a $\leq_{\mathbb{P}}$-decreasing sequence. We say that $(p_n)_{n \in \omega}$ is $(M, \mathbb{P})$-*generic* if:

(1) $p_n \in M$ for all $n \in \omega$ and

(2) for all $D \in M$ open dense subset of $\mathbb{P}$, there is $n \in \omega$ such that $p_n \in D$.

Definition 6.13. Let $\mathbb{P}$ be a forcing and $S \subseteq \omega_1$ stationary. We say that $\mathbb{P}$ is $\sigma(S)$-*closed* if for all κ large enough cardinal and all $M \preceq \mathsf{H}(\kappa)$ countable such that $\mathbb{P} \in M$ and $M \cap \omega_1 \in S$, we have that every $(M, \mathbb{P})$-generic sequence has a lower bound.

Theorem 6.14. *If $\mathbb{P}$ is $\sigma(S)$-closed and $V \models \Diamond(S)$, then $\Vdash_{\mathbb{P}} ``\Diamond(S)"$.*

Proof. Fix a sequence $\langle (M_\alpha, p_\alpha, \dot{X}_\alpha) \mid \alpha \in S \cap \mathrm{LIM}(\omega_1) \rangle$ as in Lemma 6.11. We want to define a sequence $\mathcal{D} = \{D_\alpha \subseteq \alpha \mid \alpha \in S\}$ that will be a witness of $\Diamond(S)$ in $V[G]$. Let $\alpha \in S$; in case

$M_\alpha \cap \omega_1 \neq \alpha$, let $D_\alpha = \emptyset$. Now, fix $\alpha \in S$ such that $M_\alpha \cap \omega_1 = \alpha$ and let $\{\alpha_n \mid n \in \omega\}$ be an enumeration of α. Fix also $\{E_n \mid n \in \omega\}$ an enumeration of all open dense subsets of $\mathbb{P}$ in M_α.

Now, let us construct a sequence $\{q_n^\alpha \mid n \in \omega\}$ and the set $D_\alpha \subseteq \alpha$ as follows:

(1) $q_0^\alpha = p_\alpha$.

(2) If q_n^α has been defined, then:

    (a) If there exists $q \in \mathbb{P}$ such that:

        (i) $q \leq q_n^\alpha$,

        (ii) $q \in E_n$ and

        (iii) $q \Vdash$ "$\alpha_n \in \dot{X}_\alpha$".

    then let q_{n+1}^α any such $q \in M_\alpha$ and set $\alpha_n \in D_\alpha$.

    (b) If not, then let $q_{n+1}^\alpha \in M_\alpha$ be such that satisfies conditions (i) and (ii) and set $\alpha_n \notin D_\alpha$.

Note that this process completely determines the set $D_\alpha \subseteq \alpha$, and, moreover, D_α is in the ground model and it satisfies the following:

(I) $\alpha_n \in D_\alpha$ if and only if $q_{n+1}^\alpha \Vdash$ "$\alpha_n \in \dot{X}_\alpha$" and

(II) $\alpha_n \notin D_\alpha$ if and only if $q_{n+1}^\alpha \Vdash$ "$\alpha_n \notin \dot{X}_\alpha$".

Claim 6.15. $(D_\alpha \mid \alpha \in S)$ is a $\Diamond(S)$-sequence in $V[G]$.

Proof of the Claim. Let $\dot{X}$ be a $\mathbb{P}$-name and $p \in \mathbb{P}$ such that $p \Vdash$ "$\dot{X} \subseteq \omega_1$". We want to prove that there exist $q \leq p$ and $\alpha \in S \cap \mathrm{LIM}(\omega_1)$ such that $q \Vdash$ "$\dot{X} \cap \alpha = D_\alpha$".

By hypothesis, there exist $\alpha \in S \cap \mathrm{LIM}(\omega_1)$ and $N \preceq \mathsf{H}(\kappa)$ such that $\mathbb{P}, p, \dot{X} \in N$, $M_\alpha \cap \omega_1 = \alpha$ and $(N, \in, \mathbb{P}, \leq_\mathbb{P}, \Vdash, p, \dot{X}) \cong (M_\alpha, \in, \mathbb{P}, \leq_\mathbb{P}, \Vdash, p_\alpha, \dot{X}_\alpha)$. So let $F : M_\alpha \to N$ be the (unique) isomorphism, in particular, $F(\mathbb{P}) = \mathbb{P}$, $F(p_\alpha) = p$ and $F(\dot{X}_\alpha) = \dot{X}$. Note that since F is an isomorphism then, for all $n \in \omega$, the following conditions hold:

(A) $q_{n+1}^\alpha \Vdash$ "$\alpha_n \in \dot{X}_\alpha$" if and only if $F(q_{n+1}^\alpha) \Vdash$ "$\alpha_n \in \dot{X}$",

(B) $q_{n+1}^\alpha \Vdash$ "$\alpha_n \notin \dot{X}_\alpha$" if and only if $F(q_{n+1}^\alpha) \Vdash$ "$\alpha_n \notin \dot{X}$" and

(C) $F(q_{n+1}^\alpha) \leq F(q_n^\alpha)$.

Now, by construction, the sequence $(q_n^\alpha)_{n \in \omega}$ is M_α-generic, so $(F(q_n^\alpha))_{n \in \omega}$ is N-generic. Since $\mathbb{P}$ is $\sigma(S)$-closed, the sequence $(F(q_n^\alpha))_{n \in \omega}$ admits a lower bound r. Moreover, we have

$$r \ \leq \ F(q_0^\alpha) \ = \ F(p_\alpha) \ = \ p.$$

We claim that $r \Vdash$ "$D_\alpha = \dot{X} \cap \alpha$". To see this, note that by Conditions (I) and (A) we have:

$$\alpha_n \in D_\alpha \implies q_{n+1}^\alpha \Vdash \text{ "}\alpha_n \in \dot{X}_\alpha\text{"} \implies F(q_{n+1}^\alpha) \Vdash \text{ "}\alpha_n \in \dot{X}\text{"} \implies r \Vdash \text{ "}\alpha_n \in \dot{X}\text{"}$$

and by Conditions (II) and (B):

$$\alpha_n \notin D_\alpha \implies q_{n+1}^\alpha \Vdash \text{ "}\alpha_n \notin \dot{X}_\alpha\text{"} \implies F(q_{n+1}^\alpha) \Vdash \text{ "}\alpha_n \notin \dot{X}\text{"} \implies r \Vdash \text{ "}\alpha_n \notin \dot{X}\text{"}.$$

$$Claim \square \hspace{10em} \square$$

We want to prove that the iteration theorem for $\sigma(S)$-closed forcings, that is, that a csi of $\sigma(S)$-closed forcings is $\sigma(S)$-closed. To this end, we must first establish some preliminary results:

Lemma 6.16. *Let $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \mid \beta \in \alpha \rangle$ be a csi, κ a cardinal and $M \preceq \mathsf{H}(\kappa)$ countable. If $(p_n)_{n \in \omega}$ is a $(M, \mathbb{P}_\alpha)$-generic sequence and $\beta \in \alpha \cap M$, then $(p_n \restriction \beta)_{n \in \omega}$ is $(M, \mathbb{P}_\beta)$-generic.*

Proof. Clearly $p_{n+1} \restriction \beta \leq p_n \restriction \beta$ for all $n \in \omega$ and also $p_n \restriction \beta \in M$ since $p_n, \beta \in M$. Therefore, the only thing left to prove is that for every open dense set $E \subseteq \mathbb{P}_\beta$ with $E \in M$, there exists some $n \in \omega$ such that $p_n \restriction \beta \in E$. To see this, note that

$$\hat{E} = \{p \in \mathbb{P}_\alpha \mid p \restriction \beta \in E\} \subseteq \mathbb{P}_\alpha$$

is open dense and belongs to M, therefore $p_n \in E$ for some $n \in \omega$, thus $p_n \restriction \beta \in E$. $\qquad\square$

Lemma 6.17. *Let $\mathbb{P}$ be a $\sigma(S)$-closed forcing such that $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is $\sigma(S)$-closed". Let $M \preceq \mathsf{H}(\kappa)$ be a countable submodel such that $\mathbb{P} * \dot{\mathbb{Q}} \in M$ and $M \cap \omega_1 \in S$. If $((p_n, \dot{q}_n))_{n \in \omega}$ is a $(M, \mathbb{P} * \dot{\mathbb{Q}})$-generic sequence and $p \in \mathbb{P}$ is such that $p \leq p_n$ for each $n \in \omega$, then there exists $\dot{q} \in \dot{\mathbb{Q}}$ such that $(p, \dot{q}) \leq (p_n, q_n)$ for all $n \in \omega$.*

Proof. Clearly, $p \Vdash$ "$q_{n+1} \leq q_n$" for all $n \in \omega$; we claim that p actually forces something stronger:

Claim 6.18. $p \Vdash$ "$(\dot{q}_n)_{n \in \omega}$ is $M[\dot{G}]$-generic", where $\dot{G}$ is a $\mathbb{P}$-name for the generic filter for $\mathbb{P}$.

Proof of the Claim. Let $\dot{D} \in M$ be a $\mathbb{P}$-name for an open dense subset of $\dot{\mathbb{Q}}$. Consider

$$E := \{(r, \dot{s}) \in \mathbb{P} * \dot{\mathbb{Q}} \mid r \Vdash \text{"}\dot{s} \in \dot{D}\text{"}\}.$$

It is straightforward to check that E is open dense, and $E \in M$ because $\dot{D} \in M$. Hence, there is some $n \in \omega$ such that $(p_n, \dot{q}_n) \in E$. Since $p \leq p_n$, we conclude $p \Vdash$ "$\dot{q}_n \in \dot{D}$". $\qquad\qquad{}_{Claim}\square$

Therefore,

$$p \Vdash \text{"}\left((\dot{q}_n)_{n \in \omega} \text{ is } M[\dot{G}]\text{-generic}\right) \wedge \left(\dot{\mathbb{Q}} \text{ is } \sigma(S)\text{-closed}\right)\text{"},$$

and consequently,

$$p \Vdash \text{"}\exists \dot{q} \in \dot{\mathbb{Q}} \left(\forall n \in \omega, \ \dot{q} \leq \dot{q}_n\right)\text{"}.$$

Thus, (p, q) is the desired condition. $\qquad\square$

The following result is the analogue of the Proper Iteration Lemma [26, Lemma 3.3H, p. 115] for $\sigma(S)$-closed forcings.

Proposition 6.19. *Let $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \mid \beta \in \alpha \rangle$ be a csi of $\sigma(S)$-closed forcings, κ be a large enough cardinal, $M \preceq \mathsf{H}(\kappa)$ be a countable submodel such that $M \cap \omega_1 \in S$ and let $\gamma \in M \cap \alpha$. If $(p_n)_{n \in \omega}$ is a $(M, \mathbb{P}_\alpha)$-generic sequence and $p \in \mathbb{P}_\gamma$ is such that $p \leq p_n \restriction \gamma$ for each $n \in \omega$, then there exists $\bar{p} \in P_\alpha$ such that $\bar{p} \restriction \gamma = p$ and $\bar{p} \leq p_n$ and for all $n \in \omega$.*

Proof. Suppose that the result is true for all $\beta < \alpha$ and let $(p_n)_{n \in \omega}$ be a $(M, \mathbb{P}_\alpha)$ generic sequence, $\gamma \in M \cap \alpha$ and $p \in \mathbb{P}_\gamma$ such that $p \leq p_n \restriction \gamma$ for all $n \in \omega$.

Case $\alpha = \beta + 1$: If $\gamma = \beta$, then the result follows directly from Lemma 6.17, so assume that $\gamma < \beta$. By Lemma 6.16 the sequence $(p_n \restriction \beta)_{n \in \omega}$ is $(M, \mathbb{P}_\beta)$-generic and thus, by the inductive hypothesis, there is $\bar{p} \in \mathbb{P}_\beta$ such that:

(1) $\bar{p} \restriction \gamma = p$ and

(2) $\bar{p} \leq p_n \upharpoonright \beta$ for all $n \in \omega$.

Now, since $\mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\alpha$ and each p_n is of the form $(p_n \upharpoonright \beta, \dot{q}_n)$ where $\dot{q}_n \in \dot{\mathbb{Q}}_\alpha$, we can apply Lemma 6.17 to this sequence and condition $\bar{p}$ and obtain a condition $\hat{p} \in \mathbb{P}_\alpha$ such that:

(i) $\hat{p} \upharpoonright \beta = \bar{p}$ and

(ii) $\hat{p} \leq p_n$ for all $n \in \omega$.

In particular, condition (i), implies that $\hat{p} \upharpoonright \gamma = \bar{p} \upharpoonright \gamma = p$.

Case α is a limit ordinal: Call $\beta := \bigcup(\alpha \cap M)$ and let $(\alpha_n)_{n \in \omega} \subseteq M$ increasing and cofinal in β such that $\alpha_0 = \gamma$. By recursion we will construct a sequence $(q_m)_{m \in \omega}$ such that $q_0 = p$ and for all $m \in \omega$:

(1) $q_m \in \mathbb{P}_{\alpha_m}$,

(2) $q_m \leq p_n \upharpoonright \alpha_m$ for all $n \in \omega$ and

(3) $q_{m+1} \upharpoonright \alpha_m = q_m$.

Clearly q_0 satisfies the three conditions, so suppose that we have already constructed $q_m \in \mathbb{P}_{\alpha_m}$. By Lemma 6.16, we know that $(p_n \upharpoonright \alpha_{m+1})_{n \in \omega}$ is $(M, \mathbb{P}_{\alpha_{m+1}})$-generic, thus, by the inductive hypothesis there exists $q_{m+1} \in \mathbb{P}_{\alpha_{m+1}}$ such that the three conditions hold. Finally, let $q = \bigcup_{m \in \omega} q_m$, which clearly is in $\mathbb{P}_\alpha$, it is a lower bound for $(p_n)_{n \in \omega}$ and $q \upharpoonright \gamma = q_0 = p$. $\qquad\square$

Applying Proposition 6.19 to $\gamma = 0$ and p the trivial condition in $\mathbb{P}_0 = \{1\}$ we get the desired result:

Theorem 6.20. *Let $S \subseteq \omega_1$ be stationary and $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \mid \beta \in \alpha \rangle$ be a countable support iteration of $\sigma(S)$-closed forcings, then $\mathbb{P}_\alpha$ is $\sigma(S)$-closed.*

As mentioned earlier, being $\sigma(S)$-closed is, for our purposes, an important property of the forcing $\mathbb{P}(S, \overrightarrow{C})$.

Proposition 6.21. $\mathbb{P}(S, \overrightarrow{C})$ *is $\sigma(\omega_1 \setminus S)$-closed.*

Proof. Let κ be a sufficiently large cardinal and let $M \preceq \mathsf{H}(\kappa)$ be a countable elementary submodel with $M \cap \omega_1 = \alpha \in \omega_1 \setminus S$ and let $(p_n)_{n \in \omega}$ be a M-generic sequence. To obtain the result, it suffices to prove the following:

Claim 6.22. $p := \bigcup_{n \in \omega} p_n : \alpha \to 2$ and $p \in \mathbb{P}(S, \overrightarrow{C})$.

Proof of the Claim. Since each $p_n \in M$, we have $\mathrm{dom}(p_n) < \alpha$ for every $n \in \omega$. Also, for every $n, m \in \omega$, the conditions p_n and p_m are compatible, so p is a function with

$$\mathrm{dom}(p) = \bigcup_{n \in \omega} \mathrm{dom}(p_n) \leq \alpha.$$

On the other hand, for any $\beta < \alpha$, the set $D_\beta = \{q \in \mathbb{P}(S, \overrightarrow{C}) \mid \mathrm{dom}(q) > \beta\}$ is open, dense, and belongs to M since $\beta, \mathbb{P} \in M$. Hence there is some $n \in \omega$ with $p_n \in D_\beta$, so $\mathrm{dom}(p) \geq \mathrm{dom}(p_n) > \beta$, which shows $p \colon \alpha \to 2$.

To see that $p \in \mathbb{P}(S, \vec{C})$, we must show $C_\beta \nsubseteq p^{-1}(\{1\})$ for each $\beta \in S \cap \mathrm{LIM}(\omega_1) \cap (\alpha+1)$. Fix such a β. Since $\beta \in S$ and $\alpha \notin S$, we have $\beta < \alpha$, so there is $n \in \omega$ with $\beta < \mathrm{dom}(p_n)$. If $C_\beta \subseteq p^{-1}(\{1\})$, then $C_\beta \subseteq p_n^{-1}(\{1\})$, contradicting $p_n \in \mathbb{P}(S, \vec{C})$. Hence $p \in \mathbb{P}(S, \vec{C})$. $\quad\quad$ $_{Claim}\square$ $\quad\quad\quad\quad$ $\square$

The following theorem brings together some key properties of forcings of the form $\mathbb{P}(S, \vec{C})$ and their iterations—in particular, their effects on $\diamondsuit(S)$ and $\diamondsuit(\omega_1 \setminus S)$—and will be crucial for our subsequent applications.

Theorem 6.23. *Assume $V \models \mathsf{GCH}$. Then there exists a forcing notion $\mathbb{P}$ such that $\mathbb{P} \Vdash$ "$\neg\clubsuit(S)$". Moreover, if $V \models \diamondsuit(\omega_1 \setminus S)$, then $\mathbb{P} \Vdash \diamondsuit(\omega_1 \setminus S)$.*

Proof. Let $\mathbb{P} = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \mid \beta \in \omega_2 \rangle$ be a csi such that for each $\beta \in \omega_2$, we have

$$\mathbb{P}_\beta \Vdash \text{``}\dot{\mathbb{Q}}_\beta = \mathbb{P}(S, \vec{\dot{C}}) \text{ for some } \vec{\dot{C}} \text{ ladder system on } S.\text{''}$$

By a standard bookkeeping argument, we can ensure that $\mathbb{P}(S, \vec{C})$ is eventually listed for every ladder system $\vec{C}$ in S that appears in the intermediate models and thus, $\Vdash_\mathbb{P}$ "$\neg\clubsuit(S)$".

Moreover, Proposition 6.21 and Theorem 6.20 imply that $\mathbb{P}$ is $\sigma(\omega_1 \setminus S)$-closed. Therefore, Theorem 6.14 yields that $\diamondsuit(\omega_1)$ is preserved from the ground model to the extension. $\quad\quad$ $\square$

The final step in the proof of Theorem 6.1 is to prove that the forcing notion from Theorem 6.23 preserves stationary sets of trees. For this purpose, we introduce the concept of *strategically-closed in models* forcings.

Let $\mathbb{P}$ be a forcing notion. We say that a countable model M is *suitable for* $\mathbb{P}$ if $\mathbb{P} \in M$ and $M \preceq \mathsf{H}(\kappa)$, where κ is a sufficiently large cardinal.

Definition 6.24. Let $\mathbb{P}$ be a forcing and M be a suitable model for $\mathbb{P}$. The *distributivity game of $\mathbb{P}$ in M* is defined as follows:

I	$p_0 \in \mathbb{P} \cap M$		$p_1 \in \mathbb{P} \cap M\ (p_1 \le q_0)$		$\dots$
II		$q_0 \in \mathbb{P} \cap M\ (q_0 \le p_0)$		$q_1 \in \mathbb{P} \cap M\ (q_1 \le p_1)$	$\dots$

and II wins if and only if the sequence $(p_n)_{n\in\omega}$ has a lower bound in $\mathbb{P}$ (equivalently, if $(q_n)_{n\in\omega}$ has a a lower bound in $\mathbb{P}$). We denote this game by $\mathcal{DG}(\mathbb{P}, M)$.

Notation 6.25. The sequence $(p_n)_{n\in\omega}$ will be called a run of the game and $(p_i)_{i\le n}$ a partial run.

Definition 6.26. Let $\mathbb{P}$ be a forcing.

- $\mathbb{P}$ is called *strategically closed in models* (abbreviated as *scm*) if for every M suitable model for $\mathbb{P}$, Player II has a winning strategy in $\mathcal{DG}(\mathbb{P}, M)$
- $\mathbb{P}$ is called *weakly strategically closed in models* (abbreviated *wscm*) if for every M suitable model for $\mathbb{P}$, Player I does not have a winning strategy in $\mathcal{DG}(\mathbb{P}, M)$.

As mentioned at the time of defining $\mathbb{P}(S, \vec{C})$, its strategic closure in models is crucial for our arguments—in fact, recognizing this property inspired the very definition of strategic closure in models.

Proposition 6.27. *Let $S \subseteq \omega_1$ be stationary and $\vec{C}$ be a ladder system on S. Then $\mathbb{P}(S, \vec{C})$ is scm.*

Proof. Call $\mathbb{P} = \mathbb{P}(S, \vec{C})$. Let M be a suitable model for $\mathbb{P}$ and set $\delta = M \cap \omega_1$. We will describe a strategy σ for Player II in the game $\mathcal{DG}(\mathbb{P}, M)$.

Let q_0 be the first move of Player I and choose an increasing sequence $(\alpha_n)_{n < \omega}$ such that $\alpha_0 = \text{dom}(q_0)$ and $\sup_n \alpha_n = \delta$. Now, pick $b_0 \in \delta \setminus \alpha_0$ so that $b_0 \in C_\delta$ if $\delta \in S$, and arbitrarily otherwise. Define

$$\sigma(q_0): \ b_0 + 1 \to 2 \quad \text{by} \quad q_0(\gamma) = \begin{cases} p_0(\gamma), & \gamma < \alpha_0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\sigma(q_0) \in M$ since $q_0, b_0 \in M$.

Suppose $(q_0, \ldots, q_n)$ is a partial run of $\mathcal{DG}(\mathbb{P}, M)$ in which Player II has followed σ. Note that there exists a condition $r \in \mathbb{P}$ such that $r \le q_n$ and $\text{dom}(r) > \alpha_n$, therefore, by elementarity and since $q_n \in M$, there is also $r' \in M$ with these properties, so set

$$\sigma(q_0, \ldots, q_n) = r'.$$

This completes the description of σ.

To see that σ is a winning strategy, consider any full run $(q_n)_{n < \omega}$ where Player II played according to σ. Let $q = \bigcup_{n < \omega} q_n$ and note that:

(1) q is a function,

(2) $\text{dom}(q) = \bigcup_{n \in \omega} \text{dom}(q_n)$ and each $\text{dom}(q_n) < \delta$ as $q_n \in M$, thus $\text{dom}(q) \le \delta$ and

(3) If $n \in \omega$, then $\text{dom}(q) \ge \text{dom}(q_{n+1}) \ge \text{dom}(\sigma(q_0, \ldots, q_n)) > \alpha_n/$

Thus $q: \delta \to 2$. To see that $q \in \mathbb{P}$, we must show $C_\beta \not\subseteq q^{-1}(\{1\})$ for each $\beta \in S \cap \text{LIM}(\omega_1) \cap (\delta + 1)$. Fix such a β. If $\beta = \delta$ then we are done since $C_\delta \not\subseteq q^{-1}(\{1\})$ as $q(b_0) = (\sigma(q_0))(b_0) = 0$. If $\beta < \alpha$, there is $n \in \omega$ with $\beta < \text{dom}(q_n)$, so, if $C_\beta \subseteq q^{-1}(\{1\})$, then $C_\beta \subseteq q_n^{-1}(\{1\})$, contradicting $q_n \in \mathbb{P}(S, \vec{C})$. Hence $q \in \mathbb{P}$, so σ is winning. $\square$

An important feature of weak strategic closure in models is that it strengthens total properness:

Theorem 6.28. *If $\mathbb{P}$ is a wscm forcing, then $\mathbb{P}$ is ω-distributive and proper.*

Proof. To see that $\mathbb{P}$ is ω-distributive, let $p \in \mathbb{P}$ and let $\dot{f}$ be a $\mathbb{P}$-name with $p \Vdash$ "$\dot{f}: \omega \longrightarrow \text{ORD}$"; we must produce $q \le p$ and a ground-model function $g: \omega \to \text{ORD}$ such that $q \Vdash$ "$\dot{f} = \check{g}$".

Fix a countable elementary submodel $M \prec H(\kappa)$ (for large enough κ) with $\mathbb{P}, p, \vec{C}, \dot{f} \in M$. We define a strategy σ for Player I in the game $\mathcal{DG}(\mathbb{P}, M)$ as follows:

- $\sigma(\emptyset) = p$.
- If Player II's first $n + 1$ moves are $(q_0, \ldots, q_n)$, note that there exists a condition $r \le q_n$ such that r decides the value of $\dot{f}(n)$. By elementarity, as $q_n, \dot{f}, \mathbb{P} \in M$, there is such an $r' \in M$, so set $\sigma(q_0, \ldots, q_n) = r'$.

Since $\mathbb{P}$ is wscm, Player I's strategy σ cannot be winning. Hence there is a complete run of the form $(p_n)_{n < \omega}$ with

(a) $p_0 = p$,

(b) each p_{n+1} decides $\dot{f}(n)$, and

(c) the sequence $(p_n)_{n \in \omega}$ has a common lower bound $q \in \mathbb{P}$.

By (b) and (c), q decides every $\dot{f}(n)$. Define $g \colon \omega \to \mathrm{ORD}$ in the ground model by

$$g(n) = \alpha \quad \text{if} \quad q \Vdash \text{“}\dot{f}(n) = \alpha\text{”}.$$

Then clearly $q \Vdash \text{“}\dot{f} = \check{g}\text{”}$, so $\mathbb{P}$ is ω-distributive.

To see that $\mathbb{P}$ is proper, let N be a suitable model for $\mathbb{P}$ and $p \in \mathbb{P} \cap N$. We must find a condition $q \leq p$ that is $(N, \mathbb{P})$-generic. Enumerate the dense open sets in N as $\{D_n \mid n < \omega\}$. Define a strategy ρ for Player I in the game $\mathcal{DG}(\mathbb{P}, N)$ by:

- $\rho(\emptyset) = p$.
- If Player II's first $n + 1$ moves are $(q_0, \ldots, q_n)$, note that there exists a condition $r \leq q_n$ with $r \in D_n$. By elementarity there is such an $r' \in N$, and we set $\rho(q_0, \ldots, q_n) = r'$.

Since $\mathbb{P}$ is wscm, ρ cannot be winning. Thus there is a full run $(p_n)_{n<\omega}$ satisfying:

(a) $p_0 = p$,

(b) $p_{n+1} \in D_n \cap N$ for each n, and

(c) the sequence (p_n) has a common lower bound $q \in \mathbb{P}$.

Then $q \leq p$ meets every open dense set in N, in particular q is $(N, \mathbb{P})$-generic. $\qquad\square$

As we said before, from Proposition 6.27 and Theorem 6.28 we get that $\mathbb{P}(S, \vec{C})$ is totally proper. Another important property of wscm forcings is that they preserve stationary subsets of trees:

Theorem 6.29. *Suppose that T is a nonspecial ω_1-tree, $X \subseteq T$ satisfies $X \notin NS^T$ and $\mathbb{P}$ is a wscm forcing. Then $\mathbb{P} \Vdash$ “$X \notin NS^T$”. In particular $\mathbb{P} \Vdash$ “T is nonspecial ”.*

Proof. Let $p \in \mathbb{P}$ and let $(\dot{A}_t, \dot{g}_t \mid t \in T)$ be a collection of $\mathbb{P}$-names such that

$$p \Vdash \text{“}\dot{g}_t \colon \dot{A}_t \to \omega \text{ is a specializing function”}$$

for every $t \in T$. We want to prove that there is some $q \leq p$ and $w \in X$ such that $q \Vdash$ “$w \notin \nabla_{t \in T} \dot{A}_t$”.

For every $q \leq p$, $t \in T$, and $n \in \omega$, consider the set

$$C(q, t, n) = \left\{ s \in T \;\middle|\; \Big(\forall z < s(q \Vdash \text{“}z \in \dot{A}_t \to \dot{g}_t(z) \neq n\text{”})\Big) \wedge \Big(\exists r \leq q \big(r \Vdash \text{“}s \in \dot{A}_t \wedge \dot{g}_t(s) = n\text{”}\big)\Big) \right\}.$$

Claim 6.30. For every $q \leq p$, $t \in T$, and $n \in \omega$, the set $C(q, t, n)$ is an antichain.

Proof of the Claim. Suppose that $s_0, s_1 \in C(q, t, n)$ with $s_0 < s_1$. Since $s_1 \in C(q, t, n)$, we have that $q \Vdash$ “$z \in \dot{A}_t \to \dot{g}_t(z) \neq n$”, for all $z < s_1$, in particular:

$$q \Vdash \text{“}s_0 \in \dot{A}_t \to \dot{g}_t(s_0) \neq n\text{”}.$$

On the other hand, since $s_0 \in C(q, t, n)$, there exists some $r \leq q$ such that

$$r \Vdash \text{“}s_0 \in \dot{A}_t \wedge \dot{g}_t(s_0) = n\text{”}.$$

Since $r \leq q$, it follows that

$$r \Vdash \text{“}(s_0 \in \dot{A}_t \to \dot{g}_t(s_0) \neq n) \wedge (s_0 \in \dot{A}_t \wedge \dot{g}_t(s_0) = n)\text{”},$$

which is impossible. $\qquad\qquad\qquad Claim\square$

Now, let $(M_\alpha \mid \alpha \in \omega_1)$ be a continuous $\in$-chain of countable elementary submodels of some $\mathsf{H}(\kappa)$, for a sufficiently large κ, such that $\mathbb{P}$, T, X, $(\dot{A}_t, \dot{g}_t \mid t \in T)$ and p belong to M_0. For every $t \in T$, let $\alpha(t) \in \omega_1$ be minimal such that $t \in M_\alpha$, and consider

$$B_t = \bigcup \{ C(q, z, n) \mid q \in M_{\alpha(t)}, \ z \leq t, \text{ and } n \in \omega \}.$$

Note that B_t is a countable union of antichains, so it is special. Let $B := \nabla_{t \in T} B_t$.

Consider the set $C := \{ M_\alpha \cap \omega_1 \mid \alpha \in \mathrm{LIM}(\omega_1) \}$, which is a club in ω_1. Since $T \upharpoonright C \in (NS^T)^*$, $X \notin NS^T$, and $B \in NS^T$, there is some

$$w \in \big((T \upharpoonright C) \cap X \big) \setminus B.$$

Consider an enumeration $\{ (t_i, n_i) \mid i \in \omega \}$ of the set $\{ (t, n) \mid t < w \land n \in \omega \}$ and let $\beta \in \mathrm{LIM}(\omega_1)$ be such that $\mathrm{ht}(w) = M_\beta \cap \omega_1 := \delta_\beta$.

We will construct a strategy σ for player I in the game $\mathcal{DG}(\mathbb{P}, M_\beta)$ such that $\sigma(\emptyset) = p_0 := p$ and for every $i \in \omega$ we have:

(3) $$p_{i+1} \Vdash \text{``} (w \notin \dot{A}_{t_i}) \lor (\dot{g}_{t_i}(w) \neq n_i) \text{''}.$$

Here, p_{i+1} denotes the $(i+1)$-st move of player I when following the strategy σ.

For this suppose that in the i-th entry of the game player II plays $q_i \in \mathbb{P} \cap M_\beta$. We know that $t_i < \omega$, i.e., $w \in t_i{\uparrow}$, so, since $w \notin B$, we have $w \notin B_{t_i} \cap t_i{\uparrow}$. In particular, $w \notin C(q_i, t_i, n_i)$, and thus one of the following holds:

(I) Exists $z < w$ such that $q_i \nVdash \text{``} z \in \dot{A}_{t_i} \to \dot{g}_{t_i}(z) \neq n_i \text{''}$, or

(II) there is no $r \leq q_i$ such that $r \Vdash \text{``} (w \in \dot{A}_{t_i}) \land (\dot{g}_{t_i}(w) = n_i) \text{''}$.

In other words, we obtain that one of the following holds:

(i) There exist $z < w$ and and $r \leq q_i$ such that $r \Vdash \text{``} (z \in \dot{A}_{t_i}) \land (\dot{g}_{t_i}(z) = n_i) \text{''}$;

(ii) $q_i \Vdash \text{``} (w \notin \dot{A}_{t_i}) \lor (\dot{g}_{t_i}(w) \neq n_i) \text{''}$.

If case (i) holds, then note the following:

- r already satisfies condition (3). Indeed, if not, there exists some $r' \leq r$ such that $r' \Vdash \text{``} (w \in \dot{A}_{t_i}) \land (\dot{g}_{t_i}(w) = z) \text{''}$, but then $r' \Vdash \text{``} (z < w) \land (\dot{g}_{t_i}(z) = \dot{g}_{t_i}(w)) \text{''}$, which is impossible because $r' \Vdash \text{``} \dot{g}_{t_i}$ is a specializing function''.
- By elementarity, as $z \in M_\beta$, we can take this $r \in M_\beta$ and declare $p_{i+1} := r$.

If case (ii) holds, then we are done by taking $p_{i+1} = q_i$. This completes the construction of the strategy σ.

As $\mathbb{P}$ is a wscm forcing, σ is not a winning strategy for Player I, thus there is some run of the game where Player I played according to σ and Player II won, that is, there exists a sequence $(p_i)_{i \in \omega}$ where $p_0 = p$, it satisfies the condition (3) and has a lower bound $\bar{p}$.

Claim 6.31. $\bar{p} \Vdash \text{``} w \notin \nabla_{t \in T} \dot{A}_t \text{''}$.

Proof of the Claim. Suppose that this is false. Then there exist $p' \leq \bar{p}$, $t < w$ and $n, i \in \omega$ such that

$$(4) \qquad\qquad p' \Vdash \text{``}(w \in \dot{A}_t) \wedge (\dot{g}_t(w) = n)\text{''}$$

and $(t, n) = (t_i, n_i)$. On the other hand, as $p' \leq p_{i+1}$ and $p_{i+1} \Vdash \text{``}(w \notin \dot{A}_t) \vee (\dot{g}_t(w) \neq n)\text{''}$, we have

$$p' \Vdash \text{``}(w \notin \dot{A}_t) \vee (\dot{g}_t(w) \neq n)\text{''},$$

which contradicts (4). $_{Claim}\square$ $\square$

We now have to prove the iteration theorem for scm forcings. As it is usual, we start by proving the case for a two-step iteration:

Lemma 6.32. *Suppose that $\mathbb{P}$ is a scm forcing and $\mathbb{P} \Vdash \text{``}\dot{\mathbb{Q}}$ is scm''. Then $\mathbb{P} * \dot{\mathbb{Q}}$ is scm.*

Proof. Fix M a suitable model for $\mathbb{P} * \dot{\mathbb{Q}}$ and $\dot{G}$ a $\mathbb{P}$-name for a generic filter for $\mathbb{P}$. Note that:

- M is also suitable for $\mathbb{P}$ and
- $\mathbb{P} \Vdash \text{``}M[\dot{G}]$ is a suitable model for $\dot{\mathbb{Q}}[\dot{G}]\text{''}$.

Let σ be a winning strategy for Player II in $\mathcal{DG}(\mathbb{P}, M)$. We will describe a strategy ρ for II in the game $\mathcal{DG}(\mathbb{P} * \dot{\mathbb{Q}}, M)$ as follows:

Let $(p_0, \dot{q}_0) \in M$ be the play of Player I in the first move. We know that there exists $\dot{\tau}$ a $\mathbb{P}$-name such that:

$$\sigma(p_0) \Vdash \text{``}\dot{\tau} \text{ is a winning strategy for II in the game } \mathcal{DG}(\dot{\mathbb{Q}}[\dot{G}], M[\dot{G}])\text{''}.$$

Thus we have that:

$$\sigma(p_0) \Vdash \text{``}\dot{\tau}(\dot{q}_0) \in M[\dot{G}] = \{\dot{a}[G] \mid \dot{a} \in M \wedge (\dot{a} \text{ is a } \mathbb{P}\text{-name for an element of } \dot{\mathbb{Q}})\}\text{''}.$$

In particular:

$$\sigma(p_0) \Vdash \text{``}\exists \dot{a} \in M(\dot{a} \text{ is a } \mathbb{P}\text{-name for an element of } \dot{\mathbb{Q}}) \wedge (\dot{a}[G] = \dot{\tau}(\dot{q}_0)[\dot{G}]).\text{''}$$

Thus we can find some $r_0 \leq \sigma(p_0)$ and a $\mathbb{P}$-name $\dot{b}_0$ such that both are in M and:

$$r_0 \Vdash \text{``}\dot{b}_0 = \dot{\tau}(\dot{q}_0).\text{''}$$

Hence we can take $\rho((p_0, \dot{q}_0))$ as $(r_0, \dot{b}_0)$, which by construction belongs to M.

In general, if $\langle (p_i, \dot{q}_i) \mid i \leq n \rangle$ is a partial run of the game $\mathcal{DG}(\mathbb{P} * \dot{\mathbb{Q}}, M)$ where II played according to ρ then, by the same arguments that before, we have:

$$\sigma(p_0, \ldots, p_n) \Vdash \text{``}\exists \dot{a} \in M(\dot{a} \text{ is a } \mathbb{P}\text{-name for an element of } \dot{\mathbb{Q}}) \wedge (\dot{a}[G] = \dot{\tau}(\dot{q}_0, \ldots, \dot{q}_n)[\dot{G}]).\text{''}$$

Thus we can find some $r_n \leq \sigma(p_0, \ldots, p_n)$ and $\dot{b}_n$ a $\mathbb{P}$-name such that both are in M and:

$$r_n \Vdash \text{``}\dot{b}_n = \dot{\tau}(\dot{q}_0, \ldots, \dot{q}_n).\text{''}$$

Thus $\rho((p_0, \dot{q}_0), \ldots, (p_n, \dot{q}_n)) := (r_n, \dot{b}_n) \in M$.

We claim that ρ is a winning strategy. For this, let $\langle (p_n, \dot{q}_n) \mid n \in \omega \rangle$ be a run of the game where II followed ρ and note the following two facts:

(1) for all $n \in \omega$ we have that

$$r_n \Vdash \text{``}(\dot{q}_0 \ldots, \dot{q}_n) \text{ is a partial run of the game } \mathcal{DG}(\dot{\mathbb{Q}}, M[\dot{G}]) \text{ where II followed } \dot{\tau}\text{''},$$

(2) $\langle p_n \mid n \in \omega \rangle$ can be seen as a run of the game $\mathcal{DG}(\mathbb{P}, M)$ where II followed the strategy σ.

By condition (2), there exists $p \in \mathbb{P}$ a lower bound of $(p_n)_{n \in \omega}$ and this way:

$$p \Vdash \text{``}\langle \dot{q}_n \mid n \in \omega \rangle \text{ is a run of the game } \mathcal{DG}(\dot{\mathbb{Q}}[\dot{G}], \dot{M}[\dot{G}]) \text{ where II followed the strategy } \dot{\tau}\text{''}$$

and also we know that $p \Vdash \text{``}\dot{\tau}$ is a winning strategy for Player II'', thus:

$$p \Vdash \text{``}\exists \dot{q} \in \dot{\mathbb{Q}}(\dot{q} \text{ is a lower bound of } (\dot{q}_n)_{n \in \omega})\text{''}.$$

This way $(p, \dot{q})$ is a lower bound of $\langle (p_n, \dot{q}_n) \mid n \in \omega \rangle)$, which proves that ρ is a winning strategy. $\square$

Recall that if $\mathbb{B}$ is a Boolean algebra and $a, b \in \mathbb{B}$, then $a \wedge b$ denotes the infimum of $\{a, b\}$, that is, the largest lower bound of a and b. Following this notation, if $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \mid \beta \in \alpha \rangle$ is a forcing iteration, $\beta_0 < \beta_1 \leq \alpha$, $p_0 \in \mathbb{P}_{\beta_0}$, $p_1 \in \mathbb{P}_{\beta_1}$ are such that $p_0 \leq p_1 \restriction \beta_0$, then we denote by $p_0 \wedge p_1$ the condition $p_0^\frown (p_1 \restriction [\beta_0, \beta_1)) \in \mathbb{P}_{\beta_1}$, which can be seen as the infimum of p_0 and p_1.

Theorem 6.33. *Let $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \mid \beta \in \alpha \rangle$ be a countable support iteration of scm forcings. Then $\mathbb{P}_\alpha$ is scm.*

Proof. By induction of α. If $\alpha = \beta + 1$ the result follows from Lemma 6.32. So assume α is a limit ordinal, M is a suitable model for $\mathbb{P}_\alpha$, $\dot{G}$ is a name for a generic filter for $\mathbb{P}_\alpha$ and $\{\alpha_m \mid m \in \omega\}$ is an enumeration of $M \cap \alpha$.

Note that M is also a suitable model for $\mathbb{P}_{\alpha_m}$ for every $m \in \omega$, so let us fix σ_m a winning strategy for II in $\mathcal{DG}(P_{\alpha_m}, M)$ (which exists by the inductive hypothesis) and also let us fix $\dot{\tau}_m$ a $\mathbb{P}_{\alpha_m}$-name such that

$$\mathbb{P}_\alpha \Vdash \text{``}\dot{\tau}_m \text{ is a winning strategy for II in } \mathcal{DG}(\dot{Q}_{\alpha_m}[\dot{G} \restriction \alpha_m], M[\dot{G} \restriction \alpha_m])\text{''}.$$

We want to show that there exists a strategy ρ that is winning for II in $\mathcal{DG}(\mathbb{P}_\alpha, M)$. To prove this, we will begin by describing some properties that our desired strategy ρ will satisfy; then, we will show that if ρ has these properties, it is winning; and finally, we will demonstrate that such a ρ can be constructed (which is the most technical part of the proof).

Promises of the strategy ρ:

We will describe the strategy ρ for II in $\mathcal{DG}(\mathbb{P}_\alpha, M)$ by considering sequences

$$\langle (s^m(n))_{n \in \omega} \mid m \in \omega \rangle, \quad \langle (q^m(n))_{n \in \omega} \mid m \in \omega \rangle, \quad \text{and} \quad \langle (\dot{b}^m(n))_{n \in \omega} \mid m \in \omega \rangle,$$

such that for every $n, m \in \omega$ we have:

$$s^m(n) \in \mathbb{P}_\alpha, \quad q^m(n) \in \mathbb{P}_{\alpha_m}, \quad \text{and} \quad \dot{b}^m(n) \in \dot{\mathbb{Q}}_{\alpha_m} \cap M.$$

Moreover, these sequences satisfy that if $(p_0, \ldots, p_m)$ is a partial run of $\mathcal{DG}(\mathbb{P}_\alpha, M)$ played according to ρ, that is,

I	p_0			p_1			$\ldots$	p_m	
II		$\rho(p_0)$			$\rho(p_0, p_1)$				$\rho(p_0, \ldots, p_m)$

then the following properties hold:

(A) For every $i \leq m$ and every $n \leq i$ we have:

(a) $\rho(p_0, \ldots, p_i) \leq s^n(i-n) \leq p_i$, and

(b) $\rho(p_0, \ldots, p_i) \restriction \alpha_n \leq q^n(i-n) \leq s^n(i-n) \restriction \alpha_n$.

(B) For every $i \leq m$ and every $k < l \leq i$ we have:

- If $\alpha_k < \alpha_l$, then $q^k(i)^\frown \dot{b}^k(i) \leq q^l((i+k)-l) \restriction (\alpha_k+1)$.
- If $\alpha_l < \alpha_k$, then $q^l(i-l)^\frown \dot{b}^l(i-l) \leq q^k(i-k) \restriction (\alpha_l+1)$.

(C) For every $n \leq m$ we have that

I	$s^n(0) \restriction \alpha_n$			$s^n(1) \restriction \alpha_n$			$\ldots$	$s^n(m-n) \restriction \alpha_n$	
II		$q^n(0)$			$q^n(1)$	$\ldots$			$q^n(m-n)$

is a partial run of the game $\mathcal{DG}(\mathbb{P}_{\alpha_n}, M)$ such that for every $i \leq m - n$:

- $q^n(i) \leq \sigma_m\big(s^n(0) \restriction \alpha_n, \ldots, s^n(i) \restriction \alpha_n\big)$, and
- $q^n(i) \Vdash \text{``}\dot{b}^n(i) = \dot{\tau}_m\big(s^n(0)(\alpha_n), \ldots, s^n(i)(\alpha_n)\big)\text{''}$.

Showing that ρ is winning: Note that if $(p_n)_{n \in \omega}$ is a run played according to the strategy ρ and $n, m \in \omega$ with $n \geq m$, then by conditions (a) and (b) we have that:

$$(*) \hspace{4cm} q^m(n-m) \leq p_n \restriction \alpha_m$$

Claim 6.34. ρ is a winning strategy.

Proof of the Claim. Suppose that $(p_n)_{n \in \omega}$ is a run of $\mathcal{DG}(\mathbb{P}_\alpha, M)$ in which II followed ρ. We will exhibit a lower bound r for $(p_n)_{n \in \omega}$ by constructing a sequence $\langle r_\beta \mid \beta \in \alpha \cap M \rangle$ such that for every $\beta \in \alpha \cap M$ we have:

(i) $r_\beta \in \mathbb{P}_\beta$,

(ii) If $\beta = \alpha_m$, then r_β is a lower bound of $(q^m(n))_{n \in \omega}$,

(iii) $r_\beta \Vdash \text{``}(\dot{b}^m(n))_{n \in \omega}$ is a run of the game $\mathcal{DG}(\dot{\mathbb{Q}}_{\alpha_m}[\dot{G} \restriction \alpha_m], M[\dot{G} \restriction \alpha_m])$, in which II followed $\dot{\tau}_m\text{''}$, and

(iv) If $\beta < \gamma$ with $\beta, \gamma \in M \cap \alpha$, then $r_\beta = r_\gamma \restriction \beta$.

Note that condition (iii) is really a consequence of condition (ii). Suppose that r_β has been constructed for every $\beta < \gamma$ in $M \cap \alpha$.

Case $\gamma = \beta + 1$: By condition (ii) and the fact that

$$r_\beta \Vdash \text{``}\dot{\tau}_m \text{ is a winning strategy for II in } \mathcal{DG}(\dot{\mathbb{Q}}_{\alpha_m}[\dot{G} \restriction \alpha_m], M[\dot{G} \restriction \alpha_m])\text{''},$$

we have

$$r_\beta \Vdash \text{``}\exists \dot{b}_m \in \dot{\mathbb{Q}}, \text{ a lower bound for } (\dot{b}^m(n))_{n \in \omega}\text{''}.$$

Then we let $r_{\beta+1} = r_\beta^\frown \dot{b}_m$.

Suppose that $\beta + 1 = \alpha_k$. We must prove that $r_{\beta+1}$ is a lower bound for $(q^k(n))_{n \in \omega}$.

Subcase $k > m$: By condition (B), we know that for all $i \in \omega$ we have

$$q^m((k-m)+i)^\frown \dot{b}^m((k-m)+i) \leq q^k(i) \restriction (\alpha_m+1) = q^k(i) \restriction \alpha_k = q^k(i).$$

On the other hand, since $r_{\beta+1} \restriction \beta = r_\beta \leq q^m((k-m)+i)$ and $r_\beta \Vdash \text{``}\dot{b}_m \leq \dot{b}^m((k-m)+i)\text{''}$, it follows that

$$r_{\beta+1} \leq q^m((k-m)+i)^\frown \dot{b}^m((k-m)+i),$$

so we are done.

Subcase $k < m$: In this case, by condition (B), for all $i \in \omega$ we have

$$q^m(i)^\frown \dot{b}^m(i) \leq q^k((m-k)+i) \restriction (\alpha_m+1) = q^k((m-k)+i) \restriction (\alpha_k) = q^k((m-k)+i).$$

Again, by the same argument as before, we obtain

$$r_{\beta+1} \leq q^m(i)^\frown \dot{b}^m(i),$$

so $r_{\beta+1} \leq q^k(n)$ for all $n \geq m - k$, which as $q^k(m-k) \leq q^k(j)$ for all $j \leq m - k$, proves that $r_{\beta+1}$ is a lower bound for $(q^k)_{n\in\omega}$.

Case γ is a limit ordinal: Let $r_\gamma = \bigcup_{\beta\in M\cap\gamma} r_\beta$ and let $k \in \omega$ be such that $\gamma = \alpha_k$. We must prove that $r_\gamma \leq q^k(n)$ for all $n \in \omega$, but it suffices to show that for all $\alpha \in \gamma \cap M$, we have $r_\gamma \restriction \alpha \leq q^k(n) \restriction \alpha$. So, let $\alpha \in \gamma \cap M$ and let $m \in \omega$ be such that $\alpha = \alpha_m$.

Subcase $k > m$: As before, by (B), we know that for all $i \in \omega$,

$$q^m((k-m)+i) \leq q^k(i) \restriction \alpha_m,$$

and by the inductive hypothesis $r_\gamma \restriction \alpha_m = r_{\alpha_m} \leq q^m((k-m)+i)$, so we are done.

Subcase $k < m$: Again, by (B), for all $i \in \omega$,

$$q^m(i) \leq q^k((m-k)+i) \restriction \alpha_m,$$

and by the inductive hypothesis $r_\gamma \restriction \alpha_m = r_{\alpha_m}$ is a lower bound of $\langle q^m(i) \mid i \in \omega \rangle$.

This completes the construction of the sequence $\langle r_\beta \mid \beta \in \alpha \cap M \rangle$. Finally, let

$$r = \bigcup_{\beta\in\alpha\cap M} r_\beta.$$

Subclaim 6.35. r is a lower bound for $(p_n)_{n\in\omega}$.

Proof of the subclaim: To see this, it suffices to prove that for every $\beta \in M \cap \alpha$ and every $n \in \omega$, we have

$$r \restriction \beta = r_\beta \leq p_n \restriction \beta.$$

Suppose that $\beta = \alpha_m$. Then, by condition (*), we know that for all $n \geq m$ we have:

$$r_\beta \leq q^m(n-m) \leq p_n \restriction \alpha_m.$$

If $n < m$, then $p_m \leq p_n$, and in particular,

$$p_m \restriction \alpha_m \leq p_n \restriction \alpha_m.$$

Thus, we conclude that $r_\beta \leq p_n \restriction \alpha_m$ for every $n \in \omega$.   *Subclaim* $\square$ *Claim* $\square$

Construction of ρ and the sequences: The construction of the strategy ρ and the sequences

$$\langle s^m(n), q^m(n), \dot{b}^m(n) \mid m, n \in \omega \rangle$$

will be such that in step $m = 0$ we construct $s^0(0)$, $q^0(0)$, and $\dot{b}^0(0)$ (and $\rho(p_0)$ of course); in step $m = 1$ we construct $s^0(1)$, $s^1(0)$, $q^0(1)$, $q^1(0)$, $\dot{b}^0(1)$, and $\dot{b}^1(0)$ (and $\rho(p_0, p_1)$); and in general, in step m we construct:

- $s^0(m), s^1(m-1), \ldots, s^m(0)$,
- $q^0(m), q^1(m-1), \ldots, q^m(0)$,
- $\dot{b}^0(m-1), \dot{b}^1(m), \ldots, \dot{b}^m(0)$ and
- $\rho(p_0, \ldots, p_m)$.

For $m = 0$, let p_0 be the first play of I, and set $s^0(0) = p_0$. Choose $q^0(0) \in \mathbb{P}_{\alpha_0}$ and $\dot{b}^0(0)$ in M such that:

- $q^0(0) \leq \sigma_0(p_0 \restriction \alpha_0)$, and
- $q^0(0) \Vdash$ "$\dot{b}^0(0) = \dot{\tau}_0(p_0(\alpha_0))$".

Let $\rho(p_0) = \left(q^0(0)^\frown \dot{b}^0(0) \right) \wedge p_0$. It is easy to see that $s^0(0)$, $q^0(0)$, $\dot{b}^0(0)$, and $\rho(p_0)$ satisfy all the conditions for $m = 0$.

Now, suppose that we have completed the construction up to the m-th move of the game, and let p_{m+1} be the $(m+1)$-st play of I in $\mathcal{DG}(\mathbb{P}_{\alpha_m}, M)$. Then the table for the game $\mathcal{DG}(\mathbb{P}_\alpha, M)$ now looks like:

I	p_0			p_1			$\ldots$	p_m			p_{m+1}
II		$\rho(p_0)$			$\rho(p_0, p_1)$		$\ldots$		$\rho(p_0, \ldots, p_m)$		

We now want to construct $\rho(p_0, \ldots, p_{m+1})$ and the following sequences:

- $s^0(m+1), s^1(m), \ldots, s^{m+1}(0)$,
- $q^0(m+1), q^1(m), \ldots, q^{m+1}(0)$, and
- $\dot{b}^0(m+1), \dot{b}^1(m), \ldots, \dot{b}^{m+1}(0)$.

For this, enumerate $\{\alpha_0, \ldots, \alpha_m, \alpha_{m+1}\}$ as $\{\gamma_0, \ldots, \gamma_{m+1}\}$ in decreasing order, that is:

$$\gamma_{m+1} < \gamma_m < \cdots < \gamma_0.$$

For every $n \leq m+1$, let $i_n \leq m+1$ be the unique index such that $\gamma_n = \alpha_{i_n}$.

Then, set

$$s^{i_0}((m+1) - i_0) := p_{m+1},$$

and, in the game $\mathcal{DG}(\mathbb{P}_{\gamma_0}, M)$, choose $q^{i_0}((m+1) - i_0)$ and $\dot{b}^{i_0}((m+1) - i_0)$ in M such that:

- $q^{i_0}((m+1) - i_0) \leq \sigma_{i_0}\left(s^{i_0}(0) \restriction \gamma_0, \ldots, s^{i_0}((m+1) - i_0) \restriction \gamma_0 \right)$, and
- $q^{i_0}((m+1) - i_0) \Vdash$ "$\dot{b}^{i_0}((m+1) - i_0) = \dot{\tau}_{i_0}\left(s^{i_0}(0)(\gamma_0), \ldots, s^{i_0}((m+1) - i_0)(\gamma_0) \right)$".

Thus, the table for the game $\mathcal{DG}(\mathbb{P}_{\alpha_{i_0}}, M)$ now appears as:

I	$s^{i_0}(0) \restriction \alpha_{i_0}$			$\ldots$	$s^{i_0}(m - i_0) \restriction \alpha_{i_0}$			$s^{i_0}((m+1) - i_0) \restriction \alpha_{i_0}$	
II		$q^{i_0}(0)$	$\ldots$			$q^{i_0}(m - i_0)$			$q^{i_0}((m+1) - i_0)$

In general, if $q^{i_n}((m+1) - i_n)$, $s^{i_n}((m+1) - i_n)$ and $\dot{b}^{i_n}((m+1) - i_n)$ have been constructed for some $n \leq m$, then we do what follows:

We know that $\gamma_{n+1} = \alpha_{i_{n+1}}$ for some $i_{n+1} \leq m+1$, then we let

$$s^{i_{n+1}}((m+1) - i_{n+1}) := \left(q^{i_n}((m+1) - i_n)^\frown \dot{b}^{i_n}((m+1) - i_n) \right) \wedge \left(s^{i_n}((m+1) - i_n) \right)$$

and we let $q^{i_{n+1}}((m+1)-i_{n+1})$ and $\dot{b}^{i_{n+1}}((m+1)-i_{n+1})$ be such that:

- $q^{i_{n+1}}((m+1)-i_{n+1}) \leq \sigma_{i_{n+1}}(s^{i_{n+1}}(0) \restriction \gamma_{n+1}, \ldots, s^{i_{n+1}}((m+1)-i_{n+1}) \restriction \gamma_{n+1})$ and
- $q^{i_{n+1}}((m+1)-i_{n+1}) \Vdash \text{``}\dot{b}^{i_{n+1}}((m+1)-i_{n+1}) = \dot{\tau}_{i_{n+1}}(s^{i_{n+1}}(0)(\gamma_{n+1}), \ldots, s^{i_{n+1}}((m+1)-i_{n+1})(\gamma_{n+1}))\text{''}$.

So the table of the game $\mathcal{DG}(\mathbb{P}_{\alpha_{i_{n+1}}, M})$ now looks like:

I	$s^{i_{n+1}}(0) \restriction \alpha_{i_{n+1}}$			$\ldots$	$s^{i_{n+1}}((m+1)-i_{n+1}) \restriction \alpha_{i_{n+1}}$	
II			$q^{i_{n+1}}(0)$	$\ldots$		$q^{i_{n+1}}((m+1)-i_{n+1})$

Finally, let be $\rho(p_0, \ldots, p_{m+1})$ as follows:

$$\left(q^{i_{m+1}}((m+1)-i_{m+1})^\frown \dot{b}^{i_{m+1}}((m+1)-i_{m+1})\right) \wedge \cdots \wedge \left(q^{i_0}((m+1)-i_0)^\frown \dot{b}^{i_0}((m+1)-i_0)\right) \wedge p_{m+1}$$

which is equal to:

$$\left(q^{i_{m+1}}((m+1)-i_{m+1})^\frown \dot{b}^{i_{m+1}}((m+1)-i_{m+1})\right) \wedge \left(s^{i_{m+1}}((m+1)-i_{m+1})\right).$$

It is not difficult to see that all the conditions are now satisfied for $m+1$. This finishes the construction of the strategy ρ and the proof of the theorem. $\qquad\square$

Finally, by Proposition 6.27, Theorem 6.33 and Theorem 6.29 we have that the forcing notion of Theorem 6.23 is scm, so if T is an ω_1-tree (in V) such that $T \restriction S \notin NS^T$ and $T \restriction (\omega_1 \setminus S) \in NS^T$, $\mathbb{P}$ preserves these properties, that is:

$$\mathbb{P} \Vdash \text{``}(T \restriction S \notin NS^T) \wedge (T \restriction (\omega_1 \setminus S) \in NS^T)\text{''}.$$

Thus, in $V[G]$ we have $(\Diamond(\omega_1 \setminus S)) \wedge (\neg\Diamond_T)$, in particular:

$$V[G] \models \Diamond \wedge (\exists T \text{ nonspecial } \omega_1\text{-tree}(\neg\Diamond_T)),$$

which proves Theorem 6.1.

7. OPEN QUESTIONS

Of course, one of the most important questions is what we can say about the diamond principle in the classical examples of nonspecial ω_1-trees without cofinal branches, namely $\sigma\mathbb{Q}$ and the tree $T(S)$:

Question. *Does $\sigma(\mathbb{Q})$ admit a nice successor partition? More generally, does $\Diamond_{\sigma\mathbb{Q}}$ hold?*

Question. *Given a bistationary $S \subseteq \omega_1$, does $\Diamond_{T(S)}$ hold?*

On a different note, Theorem 3.17 shows that a necessary condition for obtaining $\Diamond_T + \neg\Diamond_S$ is that $S \not\leq T$. A natural question is whether sufficient conditions can be given:

Question. *Given ω_1-trees S and T, are there sufficient conditions on S and T that determine when one can force $\Diamond_T$ together with $\neg\Diamond_S$?*

By part (ii) of Theorem 2.13, we have $NS_{\omega_1} \subseteq NS_T$ for every tree T of height ω_1. On the other hand, for every normal ideal $\mathcal{I}$ on ω_1 we have $NS_{\omega_1} \subseteq \mathcal{I}$. A natural question, therefore, is the following:

Question. *Given a normal ideal $\mathcal{I}$ on ω_1, is there a tree T of height ω_1 such that, for every $X \subseteq \omega_1$,*

$$X \in \mathcal{I} \quad \text{if and only if} \quad T \restriction X \in NS^T ?$$

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References

[1] J. E. Baumgartner, L. A. Harrington, and E. M. Kleinberg. Adding a closed unbounded set. *J. Symbolic Logic*, 41(2):481–482, 1976.

[2] J. E. Baumgartner, A. D. Taylor, and S. Wagon. Structural properties of ideals. *Dissertationes Math. (Rozprawy Mat.)*, 197:95, 1982.

[3] James E. Baumgartner. Iterated forcing. In *Surveys in set theory*, volume 87 of *London Math. Soc. Lecture Note Ser.*, pages 1–59. Cambridge Univ. Press, Cambridge, 1983.

[4] Ari Meir Brodsky. *A Theory of Stationary Trees and the Balanced Baumgartner-Hajnal-Todorcevic Theorem for Trees*. ProQuest LLC, Ann Arbor, MI, 2014. Thesis (Ph.D.)–University of Toronto (Canada).

[5] Ari Meir Brodsky. A theory of stationary trees and the balanced Baumgartner-Hajnal-Todorcevic theorem for trees. *Bull. Symb. Log.*, 25(2):219, 2019.

[6] Keith J. Devlin. $\aleph_1$-trees. *Ann. Math. Logic*, 13(3):267–330, 1978.

[7] Keith J. Devlin. Variations on $\diamondsuit$. *J. Symbolic Logic*, 44(1):51–58, 1979.

[8] Keith J. Devlin and Saharon Shelah. Souslin properties and tree topologies. *Proc. London Math. Soc. (3)*, 39(2):237–252, 1979.

[9] Todd Eisworth, Justin Tatch Moore, and David Milovich. Iterated forcing and the continuum hypothesis. In *Appalachian set theory 2006–2012*, volume 406 of *London Math. Soc. Lecture Note Ser.*, pages 207–244. Cambridge Univ. Press, Cambridge, 2013.

[10] G. Fodor. Eine Bemerkung zur Theorie der regressiven Funktionen. *Acta Sci. Math. (Szeged)*, 17:139–142, 1956.

[11] Osvaldo Guzmán and Michael Hrušák. MAD families and strategically bounding forcings. *Eur. J. Math.*, 8(1):309–334, 2022.

[12] Michael Hrušák. Combinatorics of filters and ideals. In *Set theory and its applications*, volume 533 of *Contemp. Math.*, pages 29–69. Amer. Math. Soc., Providence, RI, 2011.

[13] Michael Hrušák. Katětov order on Borel ideals. *Arch. Math. Logic*, 56(7-8):831–847, 2017.

[14] Michael Hrušák and Carlos Martínez Ranero. Some remarks on non-special coherent Aronszajn trees. *Acta Univ. Carolin. Math. Phys.*, 46(2):33–40, 2005.

[15] Michal Hrušák. Life in the Sacks model. *Acta Univ. Carolin. Math. Phys.*, 42(2):43–58, 2001. 29th Winter School on Abstract Analysis (Lhota nad Rohanovem/Zahrádky u České Lípy, 2001).

[16] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, millennium edition, 2003.

[17] R. Björn Jensen. The fine structure of the constructible hierarchy. *Ann. Math. Logic*, 4:229–308; erratum, ibid. 4 (1972), 443, 1972. With a section by Jack Silver.

[18] Ronald Jensen. Notes on set theory. *Unpublished Manuscript*, page ??, 1972.

[19] Miroslav Katětov. Products of filters. *Comment. Math. Univ. Carolinae*, 9:173–189, 1968.

[20] Ziemowit Kostana, Assaf Rinot, and Saharon Shelah. Diamond on Kurepa trees. http://assafrinot.com/paper/66, 2024. Submitted August 2024.

[21] Kenneth Kunen. *Set theory*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam-New York, 1980. An introduction to independence proofs.

[22] Kenneth Kunen. *Set theory*, volume 34 of *Studies in Logic (London)*. College Publications, London, 2011.

[23] Georges Kurepa. *Ensembles ordonnées et ramifiés*. NUMDAM, [place of publication not identified], 1935.

[24] Georges Kurepa. Ensembles linéaires et une classe de tableaux ramifiés (tableaux ramifiés de m. aronszajn). *Publ. Math. Univ. Belgrade*, 6:129–160, 1936.

[25] Yinhe Peng. *Characterization and transformation of Countryman lines and R-embeddable coherent trees in ZFC*. PhD thesis, PhD thesis, National University of Singapore, 2013.

[26] Saharon Shelah. *Proper and improper forcing*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.

[27] Masahiro Shioya. Diamonds on $P_\kappa \lambda$. In *Computational prospects of infinity. Part II. Presented talks*, volume 15 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 271–298. World Sci. Publ., Hackensack, NJ, 2008.

[28] Jack Silver. The independence of Kurepa's conjecture and two-cardinal conjectures in model theory. In *Axiomatic Set Theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967)*, volume XIII, Part I of *Proc. Sympos. Pure Math.*, pages 383–390. Amer. Math. Soc., Providence, RI, 1971.

[29] Franklin D. Tall. Some applications of a generalized Martin's axiom. *Topology Appl.*, 57(2-3):215–248, 1994.

[30] Stevo Todorcevic. Lipschitz maps on trees. *J. Inst. Math. Jussieu*, 6(3):527–556, 2007.

[31] Stevo Todorcevic. *Walks on ordinals and their characteristics*, volume 263 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.

[32] S. Todorčević. On a conjecture of R. Rado. *J. London Math. Soc. (2)*, 27(1):1–8, 1983.

[33] S. Todorčević. Trees and linearly ordered sets. In *Handbook of set-theoretic topology*, pages 235–293. North-Holland, Amsterdam, 1984.

[34] S. Todorčević. Representing trees as relatively compact subsets of the first Baire class. *Bull. Cl. Sci. Math. Nat. Sci. Math.*, No. 30:29–45, 2005.

[35] Stevo Todorčević. Stationary sets, trees and continuums. *Publ. Inst. Math. (Beograd) (N.S.)*, 29(43):249–262, 1981.

[36] Stevo Todorčević. Partition relations for partially ordered sets. *Acta Math.*, 155(1-2):1–25, 1985.

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